

Discrete Painlevé equations, Orthogonal Polynomials on the Unit Circle and N -recurrences for averages over $U(N)$ – P_{VI} τ -functions

P.J. Forrester and N.S. Witte

Department of Mathematics and Statistics

University of Melbourne, Victoria 3010, Australia

email: p.forrester@ms.unimelb.edu.au; n.witte@ms.unimelb.edu.au

The theory of orthogonal polynomials on the unit circle is developed for a general class of weights leading to systems of recurrence relations and derivatives of the polynomials and their associated functions, and to functional-difference equations of certain coefficient functions appearing in the theory. A natural formulation of the Riemann-Hilbert problem is presented which has as its solution the above system of orthogonal polynomials and associated functions. In particular for the case of regular semi-classical weights on the unit circle $w(z) = \prod_{j=1}^m (z - z_j(t))^{\rho_j}$, consisting of $m \in \mathbb{Z}_{>0}$ singularities, difference equations with respect to the orthogonal polynomial degree n (Laguerre-Freud equations) and differential equations with respect to the deformation variables $z_j(t)$ (Schlesinger equations) are derived completely characterising the system. It is shown in the simplest non-trivial case of $m = 3$ that quite generally and simply the difference equations are equivalent to the discrete Painlevé equation associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$ and no other. In a three way comparison with other methods employed on this problem - the Toeplitz lattice and Virasoro constraints, the isomonodromic deformation of 2×2 linear Fuchsian differential equations, and the algebraic approach based upon the affine Weyl group symmetry - it is shown all are entirely equivalent, when reduced in order by exact summation, to the above discrete Painlevé equation through explicit transformation formulae. The fundamental matrix integrals over the unitary group $U(N)$ arising in the theory are given by the generalised hypergeometric function ${}_2F_1^{(1)}$. From the general results flow a number of applications to physical models and we give the simplest, lowest order recurrence relations for the gap probabilities and moments of characteristic polynomials of the circular unitary ensemble (CUE_N) of random matrices and the diagonal spin-spin correlation function of the square lattice Ising model.

MSC(2000): 05E35, 39A05, 37F10, 33C45, 34M55

CONTENTS

1. Introduction	2
2. Orthogonal Polynomials on the Unit Circle and Riemann-Hilbert Problem	5
3. Regular Semi-classical Weights and Isomonodromic Deformations	14
4. The simplest Semi-classical Class: P_{VI} System	24

5. The τ -function Theory for P_{VI}	35
6. Applications to Physical Models	39
6.1. Random Matrix Averages	39
6.2. 2-D Ising Model	41
Acknowledgments	44
References	44

1. INTRODUCTION

With $-\pi < \theta \leq \pi$, $z_l := e^{i\theta_l}$ the unitary group $U(N)$ with Haar (uniform) measure has eigenvalue probability density function (see e.g. [18, Chapter 2])

$$(1.1) \quad \frac{1}{(2\pi)^N N!} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2.$$

Our interest is in averages over $U(N)$ of class functions $w(U)$ which have the factorization property $\prod_{l=1}^N w(z_l)$ for $\{z_1, \dots, z_N\} \in \text{Spec}(U(N))$. Introducing the Fourier components $\{w_l\}_{l=0, \pm 1, \dots}$ of the weight $w(z)$ by $w(z) = \sum_{l=-\infty}^{\infty} w_l z^l$, due to the well known identity [45]

$$(1.2) \quad \left\langle \prod_{l=1}^N w(z_l) \right\rangle_{U(N)} = \det[w_{i-j}]_{i,j=1, \dots, N},$$

we are equivalently studying Toeplitz determinants. As an explicit example consider the unitary average

$$(1.3) \quad T_N(t; \omega_1, \omega_2, \mu; \xi) := \left\langle \prod_{l=1}^N (1 - \xi \chi_{(\pi-\phi, \pi)}^{(l)}) e^{\omega_2 \theta_l} |1 + z_l|^{2\omega_1} \left(\frac{1}{t z_l}\right)^\mu (1 + t z_l)^{2\mu} \right\rangle_{U(N)} \Big|_{t=e^{i\phi}},$$

with $\chi_{(\pi-\phi, \pi)}^{(l)} = 1$ for $\theta_l \in (\pi - \phi, \pi)$ and $\chi_{(\pi-\phi, \pi)}^{(l)} = 0$ otherwise. For special choices of the parameters (1.3) occurs in a variety of problems from mathematical physics. Thus the case $(\omega_1, \omega_2, \mu) = (0, 0, 0)$ is the generating function for the probability that the interval $(\pi - \phi, \pi)$ contains exactly k eigenvalues in Dyson's circular unitary ensemble (which is equivalent to the unitary group with Haar measure), while the case $(\omega_1, \omega_2, \mu) = (1, 0, 1)$ is (apart from a simple factor) the generating function for the probability density function of the event that two eigenvalues in the circular unitary ensemble of $(N+2) \times (N+2)$ matrices are an angle ϕ apart with exactly k eigenvalues in between. The case $\xi = 2$, $\omega_2 = 0$, $\mu = \omega_1 = 1/2$ of (1.3) corresponds to the density matrix for the impenetrable Bose gas [34, 19]. Furthermore in the case $\xi = 0$ one sees that (1.3) includes as special cases

$$(1.4) \quad \left\langle \prod_{l=1}^N z_l^{1/4} |1 + z_l|^{-1/2} (1 + k^{-2} z_l)^{1/2} \right\rangle_{U(N)}, \quad 1/k^2 \leq 1,$$

$$(1.5) \quad \left\langle \prod_{l=1}^N (1 + 1/z_l)^{v'} (1 + q^2 z_l)^v \right\rangle_{U(N)}, \quad q^2 < 1.$$

The average (1.4) is equivalent to the Toeplitz determinant given by Onsager for the diagonal spin-spin correlation in the two-dimensional Ising model [42], while (1.5) occurs as a

cumulative probability density in the study of processes relating to increasing subsequences [7, 14].

In [22] the average (1.3) was characterized as a τ -function in Okamoto's Hamiltonian theory of the sixth Painlevé (P_{VI}) equation, up to a change of variables and/or multiplication by an elementary function. The τ -function $\tau(t)$ is defined to be

$$(1.6) \quad H = \frac{d}{dt} \log \tau,$$

where the Hamiltonian $H(q, p; \alpha, t)$ is a rational function of the co-ordinates and momenta q, p and of parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, ($\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$), and independent variable (deformation variable) t . The dynamics $T_t : q(0), p(0) \mapsto q(t), p(t)$ is governed by the Hamilton equations

$$(1.7) \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

where eliminating $p(t)$ gives the sixth Painlevé equation in $q(t)$. The above identification implies that the logarithmic derivative of T_N with respect to t is an auxiliary Hamiltonian for the P_{VI} system and it was shown in [22] that it satisfies a difference equation related to a particular discrete Painlevé equation with respect to increments in unit amounts of one of the parameters α (or in particular μ). One of the objectives of this paper is to derive discrete Painlevé type recurrences for T_N directly with respect to increments in N only. In the algebraic approach [20] the strategy is to choose a particular shift operator or Schlesinger transformation L constructed from compositions of fundamental reflection operators and Dynkin diagram automorphisms of the P_{VI} symmetry group $W_a(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, r_1, r_3, r_4 \rangle$ and use it to generate sequences of parameters and dynamical variables $L^n : \{\alpha, q, p, H, \tau\} \mapsto \{\alpha_n, q_n, p_n, H_n, \tau_n\}$ for $n = 0, 1, \dots$. This operator is chosen to increment the parameters such that only $N \mapsto N+1$ and thus τ_N is essentially the average (1.3). In all our applications the initial point $\alpha(N=0)$ is located on a reflection hyperplane in the space $\alpha \in \mathbb{C}^4$ and this characterises the entire sequence as classical solutions to the P_{VI} system. The initial member $\tau_0 = 1$ and the first nontrivial member τ_1 is a solution of the Gauss hypergeometric differential equation.

The discrete Painlevé equation that is fundamental in the P_{VI} system is that associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$

$$(1.8) \quad g_{n+1}g_n = t \frac{(f_n + 1 - \alpha_2)(f_n + 1 - \alpha_0 - \alpha_2)}{f_n(f_n + \alpha_3)}$$

$$(1.9) \quad f_n + f_{n-1} = -\alpha_3 + \frac{\alpha_1}{g_n - 1} + \frac{\alpha_4 t}{g_n - t},$$

where $\alpha_1 \mapsto \alpha_1 + 1, \alpha_2 \mapsto \alpha_2 - 1, \alpha_4 \mapsto \alpha_4 + 1$ as $n \mapsto n+1$. We label the particular discrete equations that arise according to the unambiguous algebraic-geometric classification of Sakai [44] by association with a degeneration of a particular rational surface into another, rather than the historical names employed (discrete fifth Painlevé equation, dP_V). In its full generality the average (1.3) was first characterised in terms of (1.8) and (1.9) in [20]. In fact this discrete Painlevé equation (but different in detail than the ones we will present below including that in [20]) for the average (1.5) is already known from the work of Borodin [12]. In contrast recurrences for the average (1.3) with $\xi = 0$ have also been obtained recently by Adler and van Moerbeke from their theory of the Toeplitz lattice

and its Virasoro algebra [2] that do not appear to relate to (1.8) and (1.9) [13]. Another of our objectives is to show that indeed the recurrences of [2] are of the discrete Painlevé type, by deriving explicit transformation formulae between them and (1.8),(1.9).

Because we are dealing with Toeplitz determinants with symbols $w(z)$ satisfying certain analytic conditions it is immediate from the theory of Szegő, Geronimus and others that orthogonal polynomial systems on the unit circle $\{\phi_n(z)\}_{n=0}^\infty$ with respect to such weights are relevant. For example if $\phi \in [0, 2\pi)$, $\omega_1, \omega_2, \mu, \xi \in \mathbb{R}$, $\xi < 1$, and $2\omega_1 > -1, 2\mu > -1$ then $w(e^{i\theta})$ where

$$w(z) = z^{-\mu-\omega} (1+z)^{2\omega_1} (1+tz)^{2\mu} \begin{cases} 1 & \theta \notin (\pi - \phi, \pi) \\ 1 - \xi & \theta \in (\pi - \phi, \pi) \end{cases},$$

is a real, positive weight defining a measure with an infinite number of points of increase and thus an orthogonal polynomial system on \mathbb{T} exists with respect to this weight by Favard's theorem. To characterise the averages (1.3) using orthogonal polynomial theory we have found it necessary to substantially develop the general theory of such systems. To a large extent this task has been completed for orthogonal polynomial systems defined on the line in the works of Bauldry [9], Bonan and Clark [11], Belmehdi and Ronveaux [10], Magnus [38, 37, 36] but had remained incomplete for those systems on the unit circle [27]. To this end we have derived closed systems of differential relations for the polynomials, their reciprocal polynomials $\{\phi_n^*(z)\}_{n=0}^\infty$, and associated functions $\{\epsilon_n(z)\}_{n=0}^\infty$, $\{\epsilon_n^*(z)\}_{n=0}^\infty$ in Proposition 2.1. In the notation of (2.52) and Corollary 2.3 let

$$Y_n(z; t) := \begin{pmatrix} \phi_n(z) & \epsilon_n(z)/w(z) \\ \phi_n^*(z) & -\epsilon_n^*(z)/w(z) \end{pmatrix},$$

and set

$$\frac{d}{dz} Y_n := A_n Y_n.$$

Entries in the matrix A_n are fixed by four coefficient functions $\Omega_n(z), \Omega_n^*(z), \Theta_n(z), \Theta_n^*(z)$ in (2.54) and complete sets of difference and functional relations for these coefficient functions are given in Proposition 2.2 and Corollary 2.2. We also formulate a 2×2 matrix Riemann-Hilbert problem in Proposition 2.3 for general classes of weights which parallels the case for orthogonal polynomials on the line [28, 16, 17, 15] and whose solution is simply related to Y_n . For our particular applications the weight (1.3) is a member of the regular semi-classical class

$$w(z) = \prod_{j=1}^m (z - z_j(t))^{\rho_j}, \quad \rho_j \in \mathbb{C},$$

with an arbitrary number m of isolated singularities located at $z_j(t)$. A key feature of such weights is that

$$\frac{1}{w(z)} \frac{d}{dz} w(z) = \frac{2V(z)}{W(z)},$$

where the polynomials $\deg V(z) < m, \deg W(z) = m$. The coefficient functions for regular semi-classical weights are polynomials of z with bounded degree $\deg \Omega_n(z) = \deg \Omega_n^*(z) = m - 1, \deg \Theta_n(z) = \deg \Theta_n^*(z) = m - 2$ (see Proposition 3.1). In addition evaluations of these functions at the singular points satisfy bilinear relations (see Proposition 3.2) which lead directly to one of the pair of coupled discrete Painlevé equations. Deformation

derivatives of the linear system of differential equations above with respect to arbitrary trajectories of the singularities are given in Proposition 3.3 which can be summarised as

$$\frac{d}{dt}Y_n := B_n Y_n = \left\{ B_\infty - \sum_{j=1}^m \frac{A_{nj}}{z - z_j} \frac{d}{dt} z_j \right\} Y_n, \quad \text{where} \quad A_n = \sum_{j=1}^m \frac{A_{nj}}{z - z_j},$$

and consequently systems of Schlesinger equations for the elements of A_{nj} (or the coefficient functions evaluated at z_j) are given in (3.67-3.69). It is quite natural that systems governed by regular semi-classical weights preserve the monodromy data of the solutions Y_n about each singularity z_j with respect to arbitrary deformations.

Another theme we wish to develop is the evaluation of the above Toeplitz determinants in terms of generalised hypergeometric functions. Given a partition $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_N)$ such that $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_N \geq 0$ one defines the generalised, multi-variable hypergeometric function through a series representation [46, 31]

$$(1.10) \quad {}_pF_q^{(1)}(a_1, \dots, a_p; b_1, \dots, b_q; t_1, \dots, t_N) = \sum_{\kappa \geq 0} \frac{[a_1]_\kappa^{(1)} \dots [a_p]_\kappa^{(1)}}{[b_1]_\kappa^{(1)} \dots [b_q]_\kappa^{(1)}} \frac{s_\kappa(t_1, \dots, t_N)}{h_\kappa}$$

for $p, q \in \mathbb{Z}_{\geq 0}$. Here the generalised Pochhammer symbols are

$$(1.11) \quad [a]_\kappa^{(1)} := \prod_{j=1}^N (a - j + 1)_{\kappa_j},$$

the hook length is

$$(1.12) \quad h_\kappa = \prod_{(i,j) \in \kappa} [a(i,j) + l(i,j) + 1],$$

where $a(i,j), l(i,j)$ are the arm and leg lengths of the (i,j) th box in the Young diagram of the partition κ , and $s_\kappa(t_1, \dots, t_N)$ is the Schur symmetric polynomial of N variables. The superscript (1) distinguishes these functions from the single variable $N = 1$ functions and also indicates that they are a special case of a more general function parameterised by an arbitrary complex number $d \neq 1$.

In Section 2 we derive systems of differential-difference and functional relations for orthogonal polynomials and associated functions on the unit circle for a general class of weights and formulate the Riemann-Hilbert problem. In Section 3 we specialise to regular semi-classical weights and derive bilinear difference equations. In addition we calculate the deformation derivatives of the orthogonal polynomial system, derive a system of Schlesinger equations and show the deformations are of the isomonodromic type. The foregoing theory is utilised in the simplest case of three singularities and the N -recurrences derived in Section 4. In Section 5 the connection of the orthogonal polynomial theory with the Okamoto τ -function theory is established. Application of our recurrences to the physical models described previously are considered in Section 6.

2. ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE AND RIEMANN-HILBERT PROBLEM

We consider a complex function for our formal weight $w(z)$, analytic in the cut complex z -plane and which possesses a Fourier expansion

$$(2.1) \quad w(z) = \sum_{k=-\infty}^{\infty} w_k z^k, \quad w_k = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \zeta^{-k},$$

where \mathbb{T} denotes the unit circle $|\zeta| = 1$, with $z = e^{i\theta}$, $\theta \in (-\pi, \pi]$. Hereafter we will assume that $z^j w(z), z^j w'(z) \in L(\mathbb{T})$ for all $j \in \mathbb{Z}$. The doubly infinite sequence $\{w_k\}_{k=-\infty}^{\infty}$ are the trigonometric moments of the distribution $w(e^{i\theta})d\theta/2\pi$ and define the trigonometric moment problem. Define the Toeplitz determinants

$$\begin{aligned}
 I_n^\epsilon[w] &:= \det \left[\int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \zeta^{\epsilon+j-k} \right]_{0 \leq j, k \leq n-1}, \\
 &= \det [w_{-\epsilon+j-k}]_{0 \leq j, k \leq n-1}, \\
 (2.2) \quad &= \frac{1}{n!} \int_{\mathbb{T}^n} \prod_{l=1}^n \frac{d\zeta_l}{2\pi i \zeta_l} w(\zeta_l) \zeta_l^\epsilon \prod_{1 \leq j < k \leq n} |\zeta_j - \zeta_k|^2,
 \end{aligned}$$

where $\epsilon = 0, \pm 1$. The last equality shows the determinants are equivalent to the CUE averages defined earlier and their generalisations. In certain circumstances the weight is real and positive, $\overline{w(z)} = w(z)$ where the bar denotes complex conjugate, for example when all the monodromy parameters are real and certain independent deformation variables $t \in \mathbb{T}$, and so the Toeplitz matrix $I_n^0[w]$ is Hermitian (see the example of the weight in section 4) but in general this will not be true.

Furthermore consider the system of orthogonal polynomials $\{\phi_n(z)\}_{n \in \mathbb{Z}_{\geq 0}}$ defined with respect to the weight $w(z)$ on the unit circle, assuming that none of the $I_n^0[w]$ vanish. This system is taken to be orthonormal

$$(2.3) \quad \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \phi_m(\zeta) \overline{\phi_n(\zeta)} = \delta_{m,n}$$

and the leading and trailing coefficients are defined by

$$(2.4) \quad \phi_n(z) = \kappa_n z^n + l_n z^{n-1} + m_n z^{n-2} + \dots + \phi_n(0) = \sum_{j=0}^n c_{n,j} z^j,$$

where κ_n is chosen to be real and positive without loss of generality. We also define the reciprocal polynomial by

$$(2.5) \quad \phi_n^*(z) := z^n \bar{\phi}_n(1/z) = \sum_{j=0}^n \bar{c}_{n,j} z^{n-j}.$$

The orthogonal polynomials are defined up to an overall factor by the orthogonality with respect to the monomials

$$(2.6) \quad \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \phi_n(\zeta) \bar{\zeta}^j = 0 \quad 0 \leq j \leq n-1,$$

whereas their reciprocal polynomials are similarly defined by

$$(2.7) \quad \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \phi_n^*(\zeta) \bar{\zeta}^j = 0 \quad 1 \leq j \leq n.$$

The system is alternatively defined by the sequence of ratios $r_n = \phi_n(0)/\kappa_n$, known as reflection coefficients because of their role in the scattering theory formulation of OPS on the unit circle, together with a companion quantity \bar{r}_n (notwithstanding the notation, only when $w(z)$ is real does \bar{r}_n equal the complex conjugate of r_n). From the Szegő theory [45] r_n and \bar{r}_n are related to the above Toeplitz determinants by

$$(2.8) \quad r_n = (-1)^n \frac{I_n^1[w]}{I_n^0[w]}, \quad \bar{r}_n = (-1)^n \frac{I_n^{-1}[w]}{I_n^0[w]}.$$

The Toeplitz determinants of central interest can then be recovered from

$$(2.9) \quad \frac{I_{n+1}^0[w]I_{n-1}^0[w]}{(I_n^0[w])^2} = 1 - r_n \bar{r}_n.$$

Further additional identities from the Szegő theory that relate the leading coefficients back to the reflection coefficients are

$$(2.10) \quad \kappa_n^2 = \kappa_{n-1}^2 + |\phi_n(0)|^2,$$

$$(2.11) \quad \frac{l_n}{\kappa_n} = \sum_{j=0}^{n-1} r_{j+1} \bar{r}_j,$$

$$(2.12) \quad \frac{m_n}{\kappa_n} = \sum_{j=1}^{n-1} r_{j+1} \left[\bar{r}_{j-1} + \bar{r}_j \frac{l_{j-1}}{\kappa_{j-1}} \right].$$

Some useful relations for the leading coefficients of the product of a monomial and an orthogonal polynomial or its derivative are

$$(2.13) \quad \begin{aligned} z\phi_n(z) &= \frac{\kappa_n}{\kappa_{n+1}}\phi_{n+1}(z) + \left(\frac{l_n}{\kappa_n} - \frac{l_{n+1}}{\kappa_{n+1}} \right) \phi_n(z) \\ &\quad + \left\{ \frac{l_n}{\kappa_{n-1}} \left(\frac{l_{n+1}}{\kappa_{n+1}} - \frac{l_n}{\kappa_n} \right) + \frac{m_n}{\kappa_{n-1}} - \frac{m_{n+1}}{\kappa_{n+1}} \frac{\kappa_n}{\kappa_{n-1}} \right\} \phi_{n-1}(z) + \pi_{n-2} \\ z^2\phi_n(z) &= \frac{\kappa_n}{\kappa_{n+2}}\phi_{n+2}(z) + \left(\frac{l_n}{\kappa_{n+1}} - \frac{l_{n+2}}{\kappa_{n+2}} \frac{\kappa_n}{\kappa_{n+1}} \right) \phi_{n+1}(z) \\ &\quad + \left\{ \frac{l_{n+1}}{\kappa_{n+1}} \left(\frac{l_{n+2}}{\kappa_{n+2}} - \frac{l_n}{\kappa_n} \right) + \frac{m_n}{\kappa_n} - \frac{m_{n+2}}{\kappa_{n+2}} \right\} \phi_n(z) + \pi_{n-1} \\ \phi'_n(z) &= n \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}(z) + \pi_{n-2} \\ z\phi'_n(z) &= n\phi_n(z) - \frac{l_n}{\kappa_{n-1}} \phi_{n-1}(z) + \pi_{n-2} \\ z^2\phi'_n(z) &= n \frac{\kappa_n}{\kappa_{n+1}} \phi_{n+1}(z) + \left\{ (n-1) \frac{l_n}{\kappa_n} - n \frac{l_{n+1}}{\kappa_{n+1}} \right\} \phi_n(z) + \pi_{n-1} \end{aligned}$$

where ' denotes the derivative with respect to z and where π_n denotes an arbitrary polynomial of the linear space of polynomials with degree at most n .

Fundamental consequences of the orthogonality condition are the mixed linear recurrence relations

$$(2.14) \quad \kappa_n \phi_{n+1}(z) = \kappa_{n+1} z \phi_n(z) + \phi_{n+1}(0) \phi_n^*(z)$$

$$(2.15) \quad \kappa_n \phi_{n+1}^*(z) = \kappa_{n+1} \phi_n^*(z) + \bar{\phi}_{n+1}(0) z \phi_n(z)$$

as well as the three-term recurrences

$$(2.16) \quad \kappa_n \phi_n(0) \phi_{n+1}(z) + \kappa_{n-1} \phi_{n+1}(0) z \phi_{n-1}(z) = [\kappa_n \phi_{n+1}(0) + \kappa_{n+1} \phi_n(0) z] \phi_n(z)$$

$$(2.17) \quad \kappa_n \bar{\phi}_n(0) \phi_{n+1}^*(z) + \kappa_{n-1} \bar{\phi}_{n+1}(0) z \phi_{n-1}^*(z) = [\kappa_n \bar{\phi}_{n+1}(0) z + \kappa_{n+1} \bar{\phi}_n(0)] \phi_n^*(z)$$

The analogue of the Christoffel-Darboux summation formula is

$$(2.18) \quad \sum_{j=0}^n \phi_j(z) \overline{\phi_j(\zeta)} = \frac{\phi_n^*(z) \overline{\phi_n^*(\zeta)} - z \bar{\zeta} \phi_n(z) \overline{\phi_n(\zeta)}}{1 - z \bar{\zeta}}$$

$$(2.19) \quad = \frac{\phi_{n+1}^*(z) \overline{\phi_{n+1}^*(\zeta)} - \phi_{n+1}(z) \overline{\phi_{n+1}(\zeta)}}{1 - z \bar{\zeta}},$$

for $z\bar{\zeta} \neq 1$.

Equations (2.16, 2.17) being second order linear difference equations admit other linearly independent solutions $\psi_n(z), \psi_n^*(z)$, and we define two such polynomial solutions, the polynomials of the second kind or associated polynomials

$$(2.20) \quad \psi_n(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) [\phi_n(\zeta) - \phi_n(z)], \quad n \geq 1, \quad \psi_0 := 1,$$

and its reciprocal polynomial $\psi_n^*(z)$. The integral formula for ψ_n^* is

$$(2.21) \quad \psi_n^*(z) := - \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) [z^n \overline{\phi_n(\zeta)} - \phi_n^*(z)], \quad n \geq 1, \quad \psi_0^* := 1.$$

A central object in the theory is the Carathéodory function, or generating function of the Toeplitz elements

$$(2.22) \quad F(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta)$$

which has the expansions inside and outside the unit circle

$$(2.23) \quad F(z) = \begin{cases} 1 + 2 \sum_{k=1}^{\infty} w_k z^k, & \text{if } |z| < 1, \\ -1 - 2 \sum_{k=1}^{\infty} w_{-k} z^{-k}, & \text{if } |z| > 1. \end{cases}$$

Having these definitions one requires two non-polynomial solutions $\epsilon_n(z), \epsilon_n^*(z)$ to the recurrences and these are constructed as linear combinations of the polynomial solutions according to

$$(2.24) \quad \begin{aligned} \epsilon_n(z) &:= \psi_n(z) + F(z)\phi_n(z) = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) \phi_n(\zeta) \\ \epsilon_n^*(z) &:= \psi_n^*(z) - F(z)\phi_n^*(z) = -z^n \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) \overline{\phi_n(\zeta)} \end{aligned}$$

$$(2.25) \quad = \frac{1}{\kappa_n} - \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) \phi_n^*(\zeta)$$

Theorem 2.1 ([24],[25],[26],[30]). $\psi_n(z), \psi_n^*(z)$ satisfy the three-term recurrence relations (2.16, 2.17) and along with $\epsilon_n(z), \epsilon_n^*(z)$ satisfy a variant of (2.14, 2.15) namely

$$(2.26) \quad \kappa_n \epsilon_{n+1}(z) = \kappa_{n+1} z \epsilon_n(z) - \phi_{n+1}(0) \epsilon_n^*(z)$$

$$(2.27) \quad \kappa_n \epsilon_{n+1}^*(z) = \kappa_{n+1} \epsilon_n^*(z) - \bar{\phi}_{n+1}(0) z \epsilon_n(z)$$

Theorem 2.2 ([24]). The Casoratians of the polynomial solutions $\phi_n, \phi_n^*, \psi_n, \psi_n^*$ are

$$(2.28) \quad \phi_{n+1}(z)\psi_n(z) - \psi_{n+1}(z)\phi_n(z) = \phi_{n+1}(z)\epsilon_n(z) - \epsilon_{n+1}(z)\phi_n(z) = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n$$

$$(2.29) \quad \phi_{n+1}^*(z)\psi_n^*(z) - \psi_{n+1}^*(z)\phi_n^*(z) = \phi_{n+1}^*(z)\epsilon_n^*(z) - \epsilon_{n+1}^*(z)\phi_n^*(z) = 2 \frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z^{n+1}$$

$$(2.30) \quad \phi_n(z)\psi_n^*(z) + \psi_n(z)\phi_n^*(z) = \phi_n(z)\epsilon_n^*(z) + \epsilon_n(z)\phi_n^*(z) = 2z^n$$

We will require the leading order terms in expansions of $\phi_n(z), \phi_n^*(z), \epsilon_n(z), \epsilon_n^*(z)$ both inside and outside the unit circle.

Corollary 2.1. *The orthogonal polynomials $\phi_n(z), \phi_n^*(z)$ have the following expansions*

$$(2.31) \quad \phi_n(z) = \begin{cases} \phi_n(0) + \frac{1}{\kappa_{n-1}}(\kappa_n \phi_{n-1}(0) + \phi_n(0) \bar{l}_{n-1})z + O(z^2) & |z| < 1 \\ \kappa_n z^n + l_n z^{n-1} + O(z^{n-2}) & |z| > 1 \end{cases}$$

$$(2.32) \quad \phi_n^*(z) = \begin{cases} \kappa_n + \bar{l}_n z + O(z^2) & |z| < 1 \\ \bar{\phi}_n(0)z^n + \frac{1}{\kappa_{n-1}}(\kappa_n \bar{\phi}_{n-1}(0) + \bar{\phi}_n(0) l_{n-1})z^{n-1} + O(z^{n-2}) & |z| > 1 \end{cases}$$

whilst the associated functions have the expansions

(2.33)

$$\frac{\kappa_n}{2} \epsilon_n(z) = \begin{cases} z^n - \frac{\bar{l}_{n+1}}{\kappa_{n+1}} z^{n+1} + O(z^{n+2}) & |z| < 1 \\ \frac{\phi_{n+1}(0)}{\kappa_{n+1}} z^{-1} + \left(\frac{\kappa_n^2}{\kappa_{n+1}^2} \frac{\phi_{n+2}(0)}{\kappa_{n+2}} - \frac{\phi_{n+1}(0)}{\kappa_{n+1}} \frac{l_{n+1}}{\kappa_{n+1}} \right) z^{-2} + O(z^{-3}) & |z| > 1 \end{cases}$$

(2.34)

$$\frac{\kappa_n}{2} \epsilon_n^*(z) = \begin{cases} \frac{\bar{\phi}_{n+1}(0)}{\kappa_{n+1}} z^{n+1} + \left(\frac{\kappa_n^2}{\kappa_{n+1}^2} \frac{\bar{\phi}_{n+2}(0)}{\kappa_{n+2}} - \frac{\bar{\phi}_{n+1}(0)}{\kappa_{n+1}} \frac{\bar{l}_{n+1}}{\kappa_{n+1}} \right) z^{n+2} + O(z^{n+3}) & |z| < 1 \\ 1 - \frac{l_{n+1}}{\kappa_{n+1}} z^{-1} + \left(\frac{l_{n+2} l_{n+1}}{\kappa_{n+2} \kappa_{n+1}} - \frac{m_{n+2}}{\kappa_{n+2}} \right) z^{-2} + O(z^{-3}) & |z| > 1 \end{cases}$$

The z -derivatives or spectral derivatives of the orthogonal polynomials in general are related to two consecutive polynomials [27] and we generalise this with the following parameterisation.

Proposition 2.1. *The derivatives of the orthogonal polynomials and associated functions are expressible as linear combinations in a related way ($' := d/dz$),*

$$(2.35) \quad W(z) \phi_n'(z) = \Theta_n(z) \phi_{n+1}(z) - (\Omega_n(z) + V(z)) \phi_n(z)$$

$$(2.36) \quad W(z) \phi_n^{*'}(z) = -\Theta_n^*(z) \phi_{n+1}^*(z) + (\Omega_n^*(z) - V(z)) \phi_n^*(z)$$

$$(2.37) \quad W(z) \epsilon_n'(z) = \Theta_n(z) \epsilon_{n+1}(z) - (\Omega_n(z) - V(z)) \epsilon_n(z)$$

$$(2.38) \quad W(z) \epsilon_n^{*'}(z) = -\Theta_n^*(z) \epsilon_{n+1}^*(z) + (\Omega_n^*(z) + V(z)) \epsilon_n^*(z)$$

with coefficient functions $W(z), V(z)$ independent of n .

Proof. The first, (2.35), was found in [27] where the coefficients were taken to be (their notation A_n, B_n should not be confused with our use of it subsequently)

$$(2.39) \quad A_n = -\frac{\kappa_{n-1} \phi_{n+1}(0)}{\kappa_n \phi_n(0)} \frac{z \Theta_n(z)}{W(z)}$$

$$(2.40) \quad B_n = \frac{1}{W(z)} \left(\Omega_n(z) + V(z) - \left[\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_{n+1}}{\kappa_n} z \right] \Theta_n(z) \right).$$

The other differential relations can be found in an analogous manner. \square

The coefficient functions $\Theta_n(z), \Omega_n(z), \Theta_n^*(z), \Omega_n^*(z)$ satisfy coupled linear recurrence relations themselves, one of which was reported in [27]. The full set are given in the following proposition.

Proposition 2.2. *The coefficient functions satisfy the coupled linear recurrence relations*

$$(2.41) \quad \Omega_n(z) + \Omega_{n-1}(z) - \left(\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_{n+1}}{\kappa_n} z \right) \Theta_n(z) + (n-1) \frac{W(z)}{z} = 0$$

$$(2.42) \quad \left(\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_{n+1}}{\kappa_n} z \right) (\Omega_{n-1}(z) - \Omega_n(z)) \\ + \frac{\kappa_n \phi_{n+2}(0)}{\kappa_{n+1} \phi_{n+1}(0)} z \Theta_{n+1}(z) - \frac{\kappa_{n-1} \phi_{n+1}(0)}{\kappa_n \phi_n(0)} z \Theta_{n-1}(z) - \frac{\phi_{n+1}(0)}{\phi_n(0)} \frac{W(z)}{z} = 0$$

$$(2.43) \quad \Omega_n^*(z) + \Omega_{n-1}^*(z) - \left(\frac{\kappa_{n+1}}{\kappa_n} + \frac{\bar{\phi}_{n+1}(0)}{\phi_n(0)} z \right) \Theta_n^*(z) - n \frac{W(z)}{z} = 0$$

$$(2.44) \quad \left(\frac{\kappa_{n+1}}{\kappa_n} + \frac{\bar{\phi}_{n+1}(0)}{\phi_n(0)} z \right) (\Omega_{n-1}^*(z) - \Omega_n^*(z)) \\ + \frac{\kappa_n \bar{\phi}_{n+2}(0)}{\kappa_{n+1} \bar{\phi}_{n+1}(0)} z \Theta_{n+1}^*(z) - \frac{\kappa_{n-1} \bar{\phi}_{n+1}(0)}{\kappa_n \bar{\phi}_n(0)} z \Theta_{n-1}^*(z) + \frac{\kappa_{n+1}}{\kappa_n} \frac{W(z)}{z} = 0$$

$$(2.45) \quad \Omega_{n+1}(z) + \Omega_n^*(z) - \left(\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} + \frac{\kappa_{n+2}}{\kappa_{n+1}} z \right) \Theta_{n+1}(z) + \frac{\kappa_{n+1}}{\kappa_n} (z \Theta_n(z) - \Theta_n^*(z)) = 0$$

$$(2.46) \quad \Omega_n(z) - \Omega_{n+1}(z) + \frac{\kappa_{n+2}}{\kappa_{n+1}} \left(z + \frac{\bar{\phi}_{n+1}(0)}{\kappa_{n+1}} \frac{\phi_{n+2}(0)}{\kappa_{n+2}} \right) \Theta_{n+1}(z) \\ + \frac{|\phi_{n+1}(0)|^2}{\kappa_{n+1} \kappa_n} \Theta_n^*(z) - \frac{\kappa_{n+1}}{\kappa_n} z \Theta_n(z) - \frac{W(z)}{z} = 0$$

$$(2.47) \quad \Omega_{n+1}^*(z) + \Omega_n(z) - \left(\frac{\kappa_{n+2}}{\kappa_{n+1}} + \frac{\bar{\phi}_{n+2}(0)}{\phi_{n+1}(0)} z \right) \Theta_{n+1}^*(z) \\ - \frac{\kappa_{n+1}}{\kappa_n} (z \Theta_n(z) - \Theta_n^*(z)) - \frac{W(z)}{z} = 0$$

$$(2.48) \quad \Omega_n^*(z) - \Omega_{n+1}^*(z) + \frac{\kappa_{n+2}}{\kappa_{n+1}} \left(1 + \frac{\phi_{n+1}(0)}{\kappa_{n+1}} \frac{\bar{\phi}_{n+2}(0)}{\kappa_{n+2}} z \right) \Theta_{n+1}^*(z) \\ + \frac{|\phi_{n+1}(0)|^2}{\kappa_{n+1} \kappa_n} z \Theta_n(z) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z) = 0$$

Proof. The first (2.41) was found in [27] by a direct evaluation of the left-hand side using integral definitions of the coefficient functions, however all of the relations follow from the compatibility of the differential relations and the recurrence relations. Thus (2.41,2.42) follow from the compatibility of (2.35) and (2.16), (2.43),(2.44) follow from (2.36) and (2.17), (2.45,2.46) follow from the combination of (2.35,2.36) and (2.14), and (2.47,2.48) follow from the combination of (2.35,2.36) and (2.15). \square

Remark 2.1. The relations given above are obviously not all independent, as for example we note that (2.41) can be derived from (2.45) with the use of (2.50) below.

Corollary 2.2. *Some additional identities satisfied by the coefficient functions are the following*

$$(2.49) \quad \frac{\phi_{n+1}(0)}{\phi_n(0)} \Theta_n(z) - \frac{\kappa_n}{\kappa_{n-1}} z \Theta_{n-1}(z) = \frac{\bar{\phi}_{n+1}(0)}{\phi_n(0)} z \Theta_n^*(z) - \frac{\kappa_n}{\kappa_{n-1}} \Theta_{n-1}^*(z)$$

$$(2.50) \quad \Omega_n^*(z) - \Omega_n(z) = -\frac{\kappa_{n+1}}{\kappa_n} (z \Theta_n(z) - \Theta_n^*(z)) + n \frac{W(z)}{z}$$

$$(2.51) \quad \Omega_n^*(z) + \Omega_n(z) = \left(1 - \frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)}{\kappa_{n+1}^2}\right) \left[\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} \Theta_{n+1}(z) + \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z)\right] + \frac{W(z)}{z}$$

For a general system of orthogonal polynomials on the unit circle the coupled recurrence relations and spectral differential relations can be reformulated in terms of first order 2×2 matrix equations (or alternatively as second order scalar equations). Here we define our matrix variables and derive such matrix relations, and this serves as an introduction to a characterisation of the general orthogonal polynomial system on the unit circle as the solution to a 2×2 matrix Riemann-Hilbert problem.

Firstly we note that the recurrence relations for the associated functions $\epsilon_n(z), \epsilon_n^*(z)$ given in (2.26,2.27) differ from those of the polynomial systems (2.14,2.15) by a reversal of the signs of $\phi_n(0), \bar{\phi}_n(0)$. We can compensate for this by constructing the 2×2 matrix

$$(2.52) \quad Y_n(z) := \begin{pmatrix} \phi_n(z) & \frac{\epsilon_n(z)}{w(z)} \\ \phi_n^*(z) & -\frac{\epsilon_n^*(z)}{w(z)} \end{pmatrix},$$

and note from (2.29) that $\det Y_n = -2z^n/w(z)$.

Corollary 2.3. *The recurrence relations for a general system of orthogonal polynomials (2.14,2.15) and their associated functions (2.26,2.27) are equivalent to the matrix recurrence*

$$(2.53) \quad Y_{n+1} := M_n Y_n = \frac{1}{\kappa_n} \begin{pmatrix} \kappa_{n+1} z & \phi_{n+1}(0) \\ \bar{\phi}_{n+1}(0) z & \kappa_{n+1} \end{pmatrix} Y_n,$$

with according to (2.10) , $\det M_n = z$.

Corollary 2.4. *The system of spectral derivatives for a general system of orthogonal polynomials and associated functions (2.35-2.38) are equivalent to the matrix differential equation*

$$(2.54) \quad \begin{aligned} Y_n' &:= A_n Y_n \\ &= \frac{1}{W(z)} \begin{pmatrix} -\left[\Omega_n(z) + V(z) - \frac{\kappa_{n+1}}{\kappa_n} z \Theta_n(z)\right] & \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z) \\ -\frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z \Theta_n^*(z) & \Omega_n^*(z) - V(z) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z) \end{pmatrix} Y_n. \end{aligned}$$

Proof. This follows from (2.35-2.38) and employing (2.14,2.15,2.26,2.27). \square

Remark 2.2. Compatibility of the relations (2.53) and (2.54) leads to

$$(2.55) \quad M_n' = A_{n+1} M_n - M_n A_n,$$

and upon examining the 11-component of this we recover the linear recurrence (2.46), the 12-component yields (2.45), whilst the 21-component gives (2.47) and the 22-component implies (2.48).

Remark 2.3. There are, in a second-order difference equation such as (2.26) or (2.27), other forms of the matrix variables and equations and these alternative forms will appear in our subsequent work. Defining

$$(2.56) \quad X_n(z; t) := \begin{pmatrix} \phi_{n+1}(z) & \frac{\epsilon_{n+1}(z)}{w(z)} \\ \phi_n(z) & \frac{\epsilon_n^*(z)}{w(z)} \end{pmatrix}, \quad X_n^*(z; t) := \begin{pmatrix} \phi_{n+1}^*(z) & \frac{\epsilon_{n+1}^*(z)}{w(z)} \\ \phi_n^*(z) & \frac{\epsilon_n^*(z)}{w(z)} \end{pmatrix},$$

we find the spectral derivatives to be

$$(2.57) \quad W(z)X_n' = \begin{pmatrix} \Omega_n(z) - V(z) + n\frac{W(z)}{z} & -\frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z\Theta_{n+1}(z) \\ \Theta_n(z) & -\Omega_n(z) - V(z) \end{pmatrix} X_n,$$

$$(2.58) \quad W(z)X_n^* = \begin{pmatrix} -\Omega_n^*(z) - V(z) + (n+1)\frac{W(z)}{z} & \frac{\kappa_n\bar{\phi}_{n+2}(0)}{\kappa_{n+1}\bar{\phi}_{n+1}(0)}z\Theta_{n+1}^*(z) \\ -\Theta_n^*(z) & \Omega_n^*(z) - V(z) \end{pmatrix} X_n^*.$$

Another system is based upon the definition

$$(2.59) \quad Z_n(z; t) := \begin{pmatrix} \phi_{n+1}(z) & \frac{\epsilon_{n+1}(z)}{w(z)} \\ \phi_n^*(z) & -\frac{\epsilon_n^*(z)}{w(z)} \end{pmatrix}, \quad Z_n^*(z; t) := \begin{pmatrix} \phi_{n+1}^*(z) & -\frac{\epsilon_{n+1}^*(z)}{w(z)} \\ \phi_n(z) & \frac{\epsilon_n(z)}{w(z)} \end{pmatrix},$$

and in this case the spectral derivatives are

$$(2.60) \quad W(z)Z_n' = \begin{pmatrix} -\Omega_n^*(z) - V(z) + \frac{\kappa_n}{\kappa_{n+1}}\Theta_n^*(z) + (n+1)\frac{W(z)}{z} & \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}^2}\Theta_{n+1}(z) \\ -\frac{\bar{\phi}_{n+1}(0)}{\kappa_{n+1}}\Theta_n^*(z) & \Omega_n^*(z) - V(z) - \frac{\kappa_n}{\kappa_{n+1}}\Theta_n^*(z) \end{pmatrix} Z_n,$$

$$(2.61) \quad W(z)Z_n^* = \begin{pmatrix} \Omega_n(z) - V(z) - \frac{\kappa_n}{\kappa_{n+1}}z\Theta_n(z) & -\frac{\kappa_n\bar{\phi}_{n+2}(0)}{\kappa_{n+1}^2}z^2\Theta_{n+1}^*(z) \\ \frac{\phi_{n+1}(0)}{\kappa_{n+1}}\Theta_n(z) & -\Omega_n(z) - V(z) + \frac{\kappa_n}{\kappa_{n+1}}z\Theta_n(z) \end{pmatrix} Z_n^*.$$

We end this section with a characterisation of a general system of orthogonal polynomials on the unit circle (and their associated functions) as a solution to a particular Riemann-Hilbert problem.

Proposition 2.3. *Consider the following Riemann-Hilbert problem for a 2×2 matrix function $Y : \mathbb{C} \rightarrow SL(2, \mathbb{C})$ defined in the following statements*

- (1) $Y(z)$ is analytic in $\{z : |z| > 1\} \cup \{z : |z| < 1\}$,

- (2) on $z \in \Sigma$ where Σ is the oriented unit circle in a counter-clockwise sense and $+$ ($-$) denote the left(right)-hand side or interior(exterior)

$$(2.62) \quad Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w(z)/z \\ 0 & 1 \end{pmatrix},$$

- (3) as $z \rightarrow \infty$

$$(2.63) \quad Y(z) = (\mathbb{I} + O(z^{-1})) \begin{pmatrix} z^n & O(z^{-2}) \\ O(z^n) & -z^{-1} \end{pmatrix},$$

- (4) as $z \rightarrow 0$

$$(2.64) \quad Y(z) = (\mathbb{I} + O(z)) \begin{pmatrix} O(1) & O(z^{n-1}) \\ O(1) & O(z^n) \end{pmatrix}.$$

It is assumed that the weight function $w(z)$ satisfies the restrictions given at the beginning of this section. Then the unique solution to this Riemann-Hilbert problem is

$$(2.65) \quad Y(z) = \begin{pmatrix} \frac{\phi_n(z)}{\kappa_n} & \frac{\epsilon_n(z)}{2\kappa_n z} \\ \kappa_n \phi_n^*(z) & -\frac{\kappa_n \epsilon_n^*(z)}{2z} \end{pmatrix}, \quad n \geq 1.$$

Proof. We firstly note from the jump condition (2.62) that Y_{11}, Y_{21} are entire $z \in \mathbb{C}$. From the 11-entry of the asymptotic condition (2.63) it is clear that $Y_{11} = \pi_n(z)$ a polynomial of degree at most n . Similarly $Y_{21} = \sigma_n(z)$ from an observation of the 21-component. From the 12- and 22-components of the jump condition we deduce

$$(2.66) \quad Y_{+12} - Y_{-12} = \frac{w(z)}{z} Y_{11}, \quad Y_{+22} - Y_{-22} = \frac{w(z)}{z} Y_{21},$$

and therefore

$$(2.67) \quad Y_{12} = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{w(\zeta) \pi_n(\zeta)}{\zeta - z}, \quad Y_{22} = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{w(\zeta) \sigma_n(\zeta)}{\zeta - z}.$$

Consider the large z expansion of Y_{12} implied by the first of these formulae

$$(2.68) \quad Y_{12} = -z^{-1} \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \pi_n(\zeta) + O(z^{-2}).$$

According to (2.63) the integral vanishes and so $\pi_n(\zeta)$ is orthogonal to the monomial ζ^0 . Now take the small z expansion

$$(2.69) \quad Y_{12} = \sum_{l=0}^{n-2} z^l \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \pi_n(\zeta) \overline{\zeta^{l+1}} + O(z^{n-1}).$$

From the 12-component of the condition (2.64) we observe that all terms in the sum vanish and we conclude the $\pi_n(\zeta)$ is orthogonal to the monomials $\zeta, \dots, \zeta^{n-1}$ and the first term which survives has the monomial ζ^n . Thus $\pi_n(z) \propto \phi_n(z)$, and from the explicit coefficient in the 11-entry of (2.63) $\pi_n(z)$ is the monic orthogonal polynomial $\phi_n(z)/\kappa_n$. We turn our attention to Y_{22} and examine the small z expansion

$$(2.70) \quad Y_{22} = \sum_{l=0}^{n-1} z^l \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \sigma_n(\zeta) \overline{\zeta^{l+1}} + O(z^n).$$

The 22-component of (2.64) tells us that all terms in the sum vanish and consequently $\sigma_n(\zeta)$ is orthogonal to all monomials ζ, \dots, ζ^n . Therefore $\sigma_n(\zeta) \propto \phi_n^*(z)$ and we can

determine the proportionality constant from the 22-component of the asymptotic formula (2.63) and comparing it with

$$(2.71) \quad Y_{22} = -z^{-1} \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \sigma_n(\zeta) + O(z^{-2}),$$

to conclude $\sigma_n(\zeta) = \kappa_n \phi_n^*(z)$. Finally we note that

$$(2.72) \quad \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{w(\zeta) \phi_n(\zeta)}{\zeta - z} = \frac{1}{2z} \epsilon_n(z), \quad \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{w(\zeta) \phi_n^*(\zeta)}{\zeta - z} = -\frac{1}{2z} \epsilon_n^*(z),$$

when $n > 0$. We also point out $\det Y = -z^{n-1}$. \square

Remark 2.4. Our original matrix solution Y_n specified by (2.52) is related to the solution of the above Riemann-Hilbert problem by

$$(2.73) \quad Y_n = \begin{pmatrix} \kappa_n & 0 \\ 0 & \frac{1}{\kappa_n} \end{pmatrix} Y \begin{pmatrix} 1 & 0 \\ 0 & \frac{2z}{w(z)} \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{1}{\kappa_n} & 0 \\ 0 & \kappa_n \end{pmatrix} Y_n \begin{pmatrix} 1 & 0 \\ 0 & \frac{w(z)}{2z} \end{pmatrix}.$$

Our formulation of the Riemann-Hilbert problem differs from those given in studies concerning orthogonal polynomial systems on the unit circle with more specialised weights [3],[6],[5], [4]. We have chosen this formulation as it is closest to that occurring for orthogonal polynomial systems of the line [15], the jump matrix is independent of the index n which only appears in the asymptotic condition and it is simply related to our matrix formulation (2.52).

3. REGULAR SEMI-CLASSICAL WEIGHTS AND ISOMONODROMIC DEFORMATIONS

All of the above results apply for a general class of weights on the unit circle but now we want to consider an additional restriction, namely the special structure of regular or generic semi-classical weights.

Definition 3.1 ([37]). The log-derivative of a regular or generic semi-classical weight function is rational in z with

$$(3.1) \quad W(z)w'(z) = 2V(z)w(z)$$

where $V(z), W(z)$ are polynomials with the following properties

- (1) $\deg(W) \geq 2$,
- (2) $\deg(V) < \deg(W)$,
- (3) the m zeros of $W(z)$, $\{z_1, z_2, \dots, z_m\}$ are distinct,
- (4) the residues $\rho_k = 2V(z_k)/W'(z_k) \notin \mathbb{Z}$.

The terminology regular refers to the connection of this definition with systems of linear second order differential equations in the complex plane which possess only isolated regular singularities, and we will see the appearance of these later. An explicit example of such a weight is that of the form $w(z) = \prod_{j=1}^m (z - z_j)^{\rho_j}$ with $z_j \neq z_k$ for $j \neq k$, which are also known as generalised Jacobi weights. In addition we will assume the polynomials defined above take the following forms

$$(3.2) \quad W(z) = \prod_{j=1}^m (z - z_j), \quad \frac{2V(z)}{W(z)} = \sum_{j=1}^m \frac{\rho_j}{z - z_j}.$$

The above definition is restrictive and has been generalised by relaxing some of the conditions in a series of works [41], [39], [40]. In these works the orthogonal polynomial systems were characterised by integral representations of semi-classical linear functionals with respect to certain paths in the complex plane.

It follows from these definitions that the Carathéodory function satisfies an inhomogeneous form of (3.1).

Lemma 3.1 ([1],[37]). *The Carathéodory function (2.22) satisfies the first order linear ordinary differential equation*

$$(3.3) \quad W(z)F'(z) = 2V(z)F(z) + U(z),$$

where $U(z)$ is a polynomial in z .

This lemma leads to the following important result.

Proposition 3.1. *The coefficient functions $\Theta_n(z)$, $\Theta_n^*(z)$, $\Omega_n(z)$, $\Omega_n^*(z)$ are polynomials in z of degree $m-2, m-2, m-1, m-1$ respectively. Specifically these have leading and trailing expansions of the form*

$$(3.4) \quad \begin{aligned} \Theta_n(z) = & (n+1 + \sum_{j=1}^m \rho_j) \frac{\kappa_n}{\kappa_{n+1}} z^{m-2} \\ & + \left\{ - \left[(n+1 + \sum_{j=1}^m \rho_j) \sum_{j=1}^m z_j - \sum_{j=1}^m \rho_j z_j \right] \frac{\kappa_n}{\kappa_{n+1}} + (n+2 + \sum_{j=1}^m \rho_j) \frac{\kappa_n^3}{\kappa_{n+1}^2 \kappa_{n+2}} \frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} \right. \\ & \left. - (n + \sum_{j=1}^m \rho_j) \frac{\phi_{n+1}(0) \bar{\phi}_n(0)}{\kappa_{n+1} \kappa_n} - 2 \frac{\kappa_n l_{n+1}}{\kappa_{n+1}^2} \right\} z^{m-3} + O(z^{m-4}) \end{aligned}$$

$$(3.5) \quad \begin{aligned} \Theta_n(z) = & [2V(0) - nW'(0)] \frac{\phi_n(0)}{\phi_{n+1}(0)} \\ & + \left\{ [2V'(0) - \tfrac{1}{2}nW''(0)] \frac{\phi_n(0)}{\phi_{n+1}(0)} + [2V(0) - (n-1)W'(0)] \frac{\kappa_n \phi_{n-1}(0)}{\kappa_{n-1} \phi_{n+1}(0)} \right. \\ & \left. + \left([(n+1)W'(0) - 2V(0)] \frac{\bar{l}_{n+1}}{\kappa_{n+1}} - [(n-1)W'(0) - 2V(0)] \frac{\bar{l}_{n-1}}{\kappa_{n+1}} \right) \frac{\phi_n(0)}{\phi_{n+1}(0)} \right\} z + O(z^2) \end{aligned}$$

$$(3.6) \quad \begin{aligned} \Theta_n^*(z) = & -(n + \sum_{j=1}^m \rho_j) \frac{\bar{\phi}_n(0)}{\bar{\phi}_{n+1}(0)} z^{m-2} \\ & + \left\{ \left[(n + \sum_{j=1}^m \rho_j) \sum_{j=1}^m z_j - \sum_{j=1}^m \rho_j z_j \right] \frac{\bar{\phi}_n(0)}{\bar{\phi}_{n+1}(0)} + (n+1 + \sum_{j=1}^m \rho_j) \frac{\bar{\phi}_n(0)}{\bar{\phi}_{n+1}(0)} \frac{l_{n+1}}{\kappa_{n+1}} \right. \\ & \left. - (n-1 + \sum_{j=1}^m \rho_j) \frac{\kappa_n \bar{\phi}_{n-1}(0) + \bar{\phi}_n(0) l_{n-1}}{\bar{\phi}_{n+1}(0)} \right\} z^{m-3} + O(z^{m-4}) \end{aligned}$$

$$(3.7) \quad \begin{aligned} \Theta_n^*(z) = & -[2V(0) - (n+1)W'(0)] \frac{\kappa_n}{\kappa_{n+1}} \\ & + \left\{ -[2V'(0) - \tfrac{1}{2}(n+1)W''(0)] \frac{\kappa_n}{\kappa_{n+1}} - [2V(0) - nW'(0)] \frac{\bar{l}_n}{\kappa_{n+1}} \right. \\ & \left. + [(n+2)W'(0) - 2V(0)] \left(\frac{\kappa_n^3}{\kappa_{n+2} \kappa_{n+1}^2} \frac{\bar{\phi}_{n+2}(0)}{\bar{\phi}_{n+1}(0)} - \frac{\kappa_n}{\kappa_{n+1}} \frac{\bar{l}_{n+1}}{\kappa_{n+1}} \right) \right\} z + O(z^2) \end{aligned}$$

$$\begin{aligned}
(3.8) \quad \Omega_n(z) &= (1 + 1/2 \sum_{j=1}^m \rho_j) z^{m-1} \\
&+ \left\{ -1/2 \left(\sum_{j=1}^m \rho_j \right) \left(\sum_{j=1}^m z_j \right) + 1/2 \sum_{j=1}^m \rho_j z_j - \sum_{j=1}^m z_j \right. \\
&\quad \left. + (n+2 + \sum_{j=1}^m \rho_j) \frac{\kappa_n^2}{\kappa_{n+2}\kappa_{n+1}} \frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} - \frac{l_{n+1}}{\kappa_{n+1}} \right\} z^{m-2} + O(z^{m-3})
\end{aligned}$$

$$\begin{aligned}
(3.9) \quad \Omega_n(z) &= V(0) - nW'(0) \\
&+ \left\{ V'(0) - \frac{1}{2}nW''(0) + \left(V(0) \frac{\kappa_n}{\kappa_{n+1}} + [V(0) - nW'(0)] \frac{\kappa_{n+1}}{\kappa_n} \right) \frac{\phi_n(0)}{\phi_{n+1}(0)} \right. \\
&\quad \left. + [V(0) - nW'(0)] \frac{\bar{l}_n}{\kappa_n} - [V(0) - (n+1)W'(0)] \frac{\bar{l}_{n+1}}{\kappa_{n+1}} \right\} z + O(z^2)
\end{aligned}$$

$$\begin{aligned}
(3.10) \quad \Omega_n^*(z) &= -1/2 \sum_{j=1}^m \rho_j z^{m-1} \\
&+ \left\{ 1/2 \left(\sum_{j=1}^m \rho_j \right) \left(\sum_{j=1}^m z_j \right) - 1/2 \sum_{j=1}^m \rho_j z_j - (n + \sum_{j=1}^m \rho_j) \frac{\kappa_n}{\kappa_{n+1}} \frac{\bar{\phi}_n(0)}{\bar{\phi}_{n+1}(0)} + \frac{l_{n+1}}{\kappa_{n+1}} \right\} z^{m-2} \\
&\quad + O(z^{m-3})
\end{aligned}$$

$$\begin{aligned}
(3.11) \quad \Omega_n^*(z) &= (n+1)W'(0) - V(0) \\
&+ \left\{ \frac{1}{2}(n+1)W''(0) - V'(0) + [(n+2)W'(0) - 2V(0)] \frac{\kappa_n^2}{\kappa_{n+2}\kappa_{n+1}} \frac{\bar{\phi}_{n+2}(0)}{\bar{\phi}_{n+1}(0)} - W'(0) \frac{\bar{l}_{n+1}}{\kappa_{n+1}} \right\} z \\
&\quad + O(z^2)
\end{aligned}$$

Proof. Following the approach of Laguerre [1] we write $F(z)$ in terms of $\phi_n(z)$, $\psi_n(z)$, $\epsilon_n(z)$ and use (3.3) to deduce

$$(3.12) \quad 0 = WF' - 2VF - U$$

$$(3.13) \quad = W \left(\frac{\epsilon_n - \psi_n}{\phi_n} \right)' - 2V \frac{\epsilon_n - \psi_n}{\phi_n} - U$$

$$(3.14) \quad = \frac{W(\psi_n \phi_n' - \phi_n \psi_n') + 2V \phi_n \psi_n - U \phi_n^2}{\phi_n^2} + W \left(\frac{\epsilon_n}{\phi_n} \right)' - 2V \frac{\epsilon_n}{\phi_n}.$$

The numerator of the first term is independent of ϵ_n , and so is a polynomial in z , and we denote this by

$$(3.15) \quad 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \Theta_n(z) = W(-\phi_n \epsilon_n' + \epsilon_n \phi_n') + 2V \phi_n \epsilon_n.$$

Given that this is a polynomial we can determine its degree and minimum power of z by utilising the expansions of ϕ_n, ϵ_n both inside and outside the unit circle, namely (2.31, 2.33). We find the degree of the right-hand side is $n+m-2$ so that $\Theta_n(z)$ is a polynomial of degree $m-2$. Developing the expansions further we arrive at (3.4). An identical argument applies to the other combination

$$(3.16) \quad 2 \frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z^{n+1} \Theta_n^*(z) = W(\phi_n^* \epsilon_n^{*'} - \epsilon_n^* \phi_n^{*'}) - 2V \phi_n^* \epsilon_n^*$$

and $\Theta_n^*(z)$ is also a polynomial of degree $m-2$ with the expansion (3.5). To establish (3.8) we utilise the other form of $\Theta_n(z)$ and (2.28) to deduce

$$(3.17) \quad W(\psi_n \phi_n' - \phi_n \psi_n') + 2V\phi_n \psi_n - U\phi_n^2 = 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \Theta_n(z)$$

$$(3.18) \quad = [\phi_{n+1} \psi_n - \psi_{n+1} \phi_n] \Theta_n(z).$$

Separating those terms with ϕ_n and ψ_n as factors we have

$$(3.19) \quad \{\Theta_n(z)\phi_{n+1} - W\phi_n' - V\phi_n\} \psi_n = \{\Theta_n(z)\psi_{n+1} - W\psi_n' + V\psi_n - U\phi_n\} \phi_n,$$

so that this polynomial contains both ϕ_n and ψ_n as factors and can be written as $\Omega_n \phi_n \psi_n$ with $\Omega_n(z)$ a polynomial of bounded degree. This latter polynomial can be defined as

$$(3.20) \quad 2 \frac{\phi_{n+1}(0)}{\kappa_n} z^n \Omega_n(z) = W(\psi_{n+1} \phi_n' - \phi_{n+1} \psi_n') + V(\phi_n \psi_{n+1} + \psi_n \phi_{n+1}) - U\phi_n \phi_{n+1}$$

$$(3.21) \quad = W(\epsilon_{n+1} \phi_n' - \phi_{n+1} \epsilon_n') + V(\phi_n \epsilon_{n+1} + \epsilon_n \phi_{n+1}).$$

Again employing the expansions (2.31, 2.33) we determine the degree of $\Omega_n(z)$ to be $m-1$ and the expansion (3.8) follows. Starting with the alternative definition of $\Theta_n^*(z)$ and (2.29)

$$(3.22) \quad W(\phi_n^* \psi_n^{*'} - \psi_n^* \phi_n^{*'}) - 2V\phi_n^* \psi_n^* - U\phi_n^{*2} = 2 \frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z^{n+1} \Theta_n^*(z)$$

$$(3.23) \quad = [\phi_{n+1}^* \psi_n^* - \psi_{n+1}^* \phi_n^*] \Theta_n^*(z).$$

and using the above argument we identify for the polynomial $\Omega_n^*(z)$

$$(3.24)$$

$$(3.25) \quad 2 \frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z^{n+1} \Omega_n^*(z) = W(-\psi_{n+1}^* \phi_n^{*'} + \phi_{n+1}^* \psi_n^{*'}) - V(\phi_n^* \psi_{n+1}^* + \psi_n^* \phi_{n+1}^*) - U\phi_n^* \phi_{n+1}^* \\ = W(-\epsilon_{n+1}^* \phi_n^{*'} + \phi_{n+1}^* \epsilon_n^{*'}) - V(\phi_n^* \epsilon_{n+1}^* + \epsilon_n^* \phi_{n+1}^*).$$

The degree of $\Omega_n^*(z)$ to be $m-1$ and has the expansion (3.10). \square

Remark 3.1. Solving for ϕ_n' and ϵ_n' between (3.15) and (3.21) leads to (2.35) and (2.37), whilst solving for $\phi_n^{*'}$ and $\epsilon_n^{*'}$ using (3.16, 3.25) yields (2.36) and (2.38).

Furthermore, in the case of a regular semi-classical weight function, the matrix $A_n(z; t)$ has the partial fraction decomposition

$$(3.26) \quad A_n(z; t) := \sum_{j=1}^m \frac{A_{nj}(t)}{z - z_j},$$

under the assumptions following (3.1) and the residue matrices are given by

$$(3.27)$$

$$A_{nj} = \frac{\rho_j}{2V(z_j)} \begin{pmatrix} -\Omega_n(z_j) - V(z_j) + \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) & \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z_j) \\ -\frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z_j \Theta_n^*(z_j) & \Omega_n^*(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j) \end{pmatrix}.$$

Using the identity (2.50) we note that $\text{Tr} A_{nj} = -\rho_j$ and $\text{Tr} A_n(z; t) = -w'(z)/w(z)$.

Bilinear residue formulae relating products of a polynomial and an associated function evaluated at a singular point will arise in the theory of the deformation derivatives later and we give a complete list of results for these.

Corollary 3.1. *Bilinear residues are related to the coefficient function residues in the following equations*

$$(3.28) \quad \phi_n(z_j)\epsilon_n(z_j) = 2\frac{\phi_{n+1}(0)}{\kappa_n}z_j^n\frac{\Theta_n(z_j)}{2V(z_j)}$$

$$(3.29) \quad \phi_n^*(z_j)\epsilon_n^*(z_j) = -2\frac{\bar{\phi}_{n+1}(0)}{\kappa_n}z_j^{n+1}\frac{\Theta_n^*(z_j)}{2V(z_j)}$$

$$(3.30) \quad \phi_{n+1}(z_j)\epsilon_n(z_j) = 2\frac{\phi_{n+1}(0)}{\kappa_n}z_j^n\frac{\Omega_n(z_j) + V(z_j)}{2V(z_j)}$$

$$(3.31) \quad \phi_n(z_j)\epsilon_{n+1}(z_j) = 2\frac{\phi_{n+1}(0)}{\kappa_n}z_j^n\frac{\Omega_n(z_j) - V(z_j)}{2V(z_j)}$$

$$(3.32) \quad \phi_n^*(z_j)\epsilon_{n+1}^*(z_j) = -2\frac{\bar{\phi}_{n+1}(0)}{\kappa_n}z_j^{n+1}\frac{\Omega_n^*(z_j) + V(z_j)}{2V(z_j)}$$

$$(3.33) \quad \phi_{n+1}^*(z_j)\epsilon_n^*(z_j) = -2\frac{\bar{\phi}_{n+1}(0)}{\kappa_n}z_j^{n+1}\frac{\Omega_n^*(z_j) - V(z_j)}{2V(z_j)}$$

$$(3.34) \quad \phi_n(z_j)\epsilon_n^*(z_j) = -\frac{z_j^n}{V(z_j)}\left[\Omega_n(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n}z_j\Theta_n(z_j)\right]$$

$$(3.35) \quad = -\frac{z_j^n}{V(z_j)}\left[\Omega_n^*(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n}\Theta_n^*(z_j)\right]$$

$$(3.36) \quad \phi_n^*(z_j)\epsilon_n(z_j) = \frac{z_j^n}{V(z_j)}\left[\Omega_n(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n}z_j\Theta_n(z_j)\right]$$

$$(3.37) \quad = \frac{z_j^n}{V(z_j)}\left[\Omega_n^*(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n}\Theta_n^*(z_j)\right].$$

Proof. These are all found by evaluating one of (3.15), (3.16), (3.21), or (3.25) at $z = z_j$ and using (2.28-2.30). \square

Remark 3.2. The initial members of the sequences of coefficient functions $\{\Theta_n\}_{n=0}^\infty$, $\{\Theta_n^*\}_{n=0}^\infty$, $\{\Omega_n\}_{n=0}^\infty$, $\{\Omega_n^*\}_{n=0}^\infty$ are given by

$$(3.38)$$

$$\Theta_0(z) = 2V(z) - U(z)$$

$$(3.39)$$

$$\Theta_1(z) = \kappa_1^2 z^2 (2V(z) - U(z)) - 2\kappa_1 \phi_1(0) z U(z) - 2\kappa_1 \phi_1(0) W(z) - \phi_1^2(0) (2V(z) + U(z))$$

$$(3.40)$$

$$\Theta_0^*(z) = -2V(z) - U(z)$$

$$(3.41)$$

$$\Theta_1^*(z) = \bar{\phi}_1^2(0) z^2 (2V(z) - U(z)) - 2\kappa_1 \bar{\phi}_1(0) z U(z) - 2\kappa_1 \bar{\phi}_1(0) W(z) - \kappa_1^2 (2V(z) + U(z))$$

$$(3.42)$$

$$\Omega_0(z) = \frac{\kappa_1}{2\phi_1(0)} z (2V(z) - U(z)) - \frac{U(z)}{2}$$

$$(3.43)$$

$$\Omega_0^*(z) = -\frac{\kappa_1}{2\bar{\phi}_1(0)z} (2V(z) + U(z)) - \frac{U(z)}{2}.$$

One can take combinations of the above functional-difference equations and construct exact differences when z is evaluated at the singular points of the weight, i.e. $W(z) = 0$. The integration of the system is given in the following proposition.

Proposition 3.2. *At all the singular points $z_j, j = 1, \dots, m$, with the exception of $z_j = 0$, the coefficient functions satisfy the bilinear identities*

$$(3.44) \quad \Omega_n^2(z_j) = \frac{\kappa_n \phi_{n+2}(0)}{\kappa_{n+1} \phi_{n+1}(0)} z_j \Theta_n(z_j) \Theta_{n+1}(z_j) + V^2(z_j)$$

$$(3.45) \quad \Omega_n^{*2}(z_j) = \frac{\kappa_n \bar{\phi}_{n+2}(0)}{\kappa_{n+1} \bar{\phi}_{n+1}(0)} z_j \Theta_n^*(z_j) \Theta_{n+1}^*(z_j) + V^2(z_j)$$

$$(3.46) \quad \left[\Omega_{n-1}(z_j) - \frac{\kappa_{n-1}^2}{\kappa_n^2} \frac{\phi_{n+1}(0)}{\phi_n(0)} \Theta_n(z_j) \right]^2 = \frac{\phi_{n+1}(0) \bar{\phi}_n(0)}{\kappa_n^2} \Theta_n(z_j) \Theta_{n-1}^*(z_j) + V^2(z_j)$$

$$(3.47) \quad \left[\Omega_{n-1}^*(z_j) - \frac{\kappa_{n-1}^2}{\kappa_n^2} \frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)} z_j \Theta_n^*(z_j) \right]^2 = \frac{\kappa_{n-1} \bar{\phi}_{n+1}(0) \phi_n(0)}{\kappa_n^3} z_j^2 \Theta_n^*(z_j) \Theta_{n-1}(z_j) + V^2(z_j)$$

$$(3.48) \quad \frac{\phi_{n+1}(0) \bar{\phi}_{n+1}(0)}{\kappa_n^2} \Theta_n(z_j) \Theta_n^*(z_j) + V^2(z_j) = \left[\Omega_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) \right]^2$$

$$(3.49) \quad = \left[\Omega_n^*(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j) \right]^2$$

First Proof. We take the first pair of identities (3.44) and (3.45) as an example for our first proof. Multiplying the Ω_n, Ω_{n-1} terms of (2.41) by the corresponding terms of (2.42), evaluated at a singular point $z = z_j$, one has an exact difference

$$(3.50) \quad \Omega_n^2(z_j) - \Omega_{n-1}^2(z_j) = \frac{\kappa_n \phi_{n+2}(0)}{\kappa_{n+1} \phi_{n+1}(0)} z_j \Theta_n(z_j) \Theta_{n+1}(z_j) - \frac{\kappa_{n-1} \phi_{n+1}(0)}{\kappa_n \phi_n(0)} z_j \Theta_{n-1}(z_j) \Theta_n(z_j),$$

assuming none of the z_j coincide with $-r_{n+1}/r_n$ for any n . Upon summing this relation the summation constant is calculated to be

$$(3.51) \quad \Omega_0^2(z_j) - \frac{\kappa_0 \phi_2(0)}{\kappa_1 \phi_1(0)} z_j \Theta_0(z_j) \Theta_1(z_j) = V^2(z_j),$$

by using the initial members of the coefficient function sequences in (3.42, 3.38, 3.39). The result is (3.44), whilst the second relation follows from an identical argument applied to (2.43, 2.44). \square

Second Proof. The three pairs of formulae (3.44, 3.45), (3.46, 3.47) and (3.48, 3.49) arise from the fact that at a singular point z_j the determinant of the matrix spectral derivative must vanish. Thus (3.44) and (3.45) express the condition that the determinant of the matrix on the right-hand sides of (2.57) and (2.58) vanish respectively. It can be shown that the same condition applied to the right-hand sides of (2.61) and (2.60) implies (3.46) and (3.47) respectively when one takes into account the identities (2.46), (2.49), (2.50) and (2.43). The last pair are a consequence of $\det(WA_n(z_j; t)) = 0$ along with the identity (2.50). \square

Third Proof. All the bilinear identities in Proposition 3.2 can be easily derived from the residue formulae (3.28-3.37) by multiplying any two of the above formulae and then factoring the resulting product in a different way. Thus (3.44) arises from multiplying (3.30) and (3.31) and then factoring the product in order to employ (3.28). Equation (3.46) comes from multiplying (3.28) and (3.29) with $n \mapsto n-1$, using the recurrences (2.15), (2.27) with $n \mapsto n-1$ to solve for $\phi_{n-1}^*(z_j), \epsilon_{n-1}^*(z_j)$ and employing (3.28) along with (3.30) and (3.31) setting $n \mapsto n-1$. Equation (3.48) is derived by multiplying (3.28)

and (3.29) and then factoring using (3.34) and (3.36). The reciprocal versions follow from similar reasoning. \square

Remark 3.3. It is clear from the first proof that the bilinear identities given in Proposition 3.2 can be straightforwardly generalised to ones that are functions of z rather than evaluated at special z values. They can be derived directly from Proposition 2.2, so apply in situations where the weights are not semi-classical, and contain additional terms with a factor of $W(z)$ and sums of products of other coefficients ranging from $j = 1, \dots, n$. However because we will have no use for such relations we refrain from writing these down.

Remark 3.4. If $z = 0$ is a singular point then the limit as $z \rightarrow 0$ may be taken in the product of (2.41, 2.42), however this does not lead to any new independent relation but simply recovers

$$\Omega_n(0) = V(0) - nW'(0).$$

We now consider the dynamics of deforming the semi-classical weight (3.2) through a t -dependence of the singular points $z_j(t)$,

$$(3.52) \quad \frac{\dot{w}}{w} = - \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z - z_j},$$

where $\dot{\cdot} := d/dt$. Given this motion of the singularities we consider the t -derivatives of the orthogonal polynomial system.

Proposition 3.3. *The deformation derivative of a semi-classical orthogonal polynomial is*

$$(3.53) \quad \dot{\phi}_n(z) = \left\{ -\frac{\dot{\kappa}_n}{\kappa_n} - \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} + 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n(z_j) \phi_n^*(z_j) \frac{z}{z - z_j} \right\} \phi_n(z) \\ - \left\{ 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n(z_j) \phi_n(z_j) \frac{1}{z - z_j} \right\} \phi_n^*(z),$$

whilst that of a reciprocal polynomial is

$$(3.54) \quad \dot{\phi}_n^*(z) = \left\{ -\frac{\dot{\kappa}_n}{\kappa_n} + 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n^*(z_j) \phi_n(z_j) \frac{1}{z - z_j} \right\} \phi_n^*(z) \\ - \left\{ 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n^*(z_j) \phi_n^*(z_j) \frac{z}{z - z_j} \right\} \phi_n(z).$$

The deformation derivative of an associated function is

$$(3.55) \quad \dot{\epsilon}_n(z) = \left\{ -\frac{\dot{\kappa}_n}{\kappa_n} - 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n^*(z_j) \phi_n(z_j) \frac{z}{z - z_j} \right\} \epsilon_n(z) \\ + \left\{ 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n(z_j) \phi_n(z_j) \frac{1}{z - z_j} \right\} \epsilon_n^*(z),$$

and that of a reciprocal associated function is

$$(3.56) \quad \epsilon_n^*(z) = \left\{ -\frac{\dot{\kappa}_n}{\kappa_n} - 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n(z_j) \phi_n^*(z_j) \frac{1}{z-z_j} \right\} \epsilon_n^*(z) \\ + \left\{ 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n^*(z_j) \phi_n^*(z_j) \frac{z}{z-z_j} \right\} \epsilon_n(z).$$

Proof. Differentiating the orthonormality condition

$$\int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \phi_n(\zeta) \overline{\phi_{n-i}(\zeta)} = \delta_{i,0}.$$

and using (3.52) we find

$$0 = \frac{\dot{\kappa}_n}{\kappa_n} \delta_{i,0} + \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \dot{\phi}_n \overline{\phi_{n-i}} - \sum_j \rho_j \dot{z}_j \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \frac{1}{\zeta - z_j} \phi_n \overline{\phi_{n-i}}, \quad i = 0, \dots, n$$

Now

$$\int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \frac{1}{\zeta - z} \phi_n(\zeta) \overline{\phi_{n-i}(\zeta)} = \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \phi_n(\zeta) \frac{\bar{\phi}_{n-i}(\zeta^{-1}) - \bar{\phi}_{n-i}(z^{-1})}{\zeta - z} \\ + \bar{\phi}_{n-i}(z^{-1}) \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \frac{\phi_n(\zeta)}{\zeta - z} \\ = -\frac{1}{z} \delta_{i,0} + \frac{\bar{\phi}_{n-i}(z^{-1})}{2z} \epsilon_n(z), \quad n > 0$$

so that

$$0 = \left(\frac{\dot{\kappa}_n}{\kappa_n} + \sum_j \rho_j \frac{\dot{z}_j}{z_j} \right) \delta_{i,0} - 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{i-n} \phi_{n-i}^*(z_j) \epsilon_n(z_j) + \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \dot{\phi}_n \overline{\phi_{n-i}}.$$

In addition we can represent $\bar{\phi}_{n-i}(z)$ as

$$\bar{\phi}_{n-i}(z) = \sum_{j=0}^n \delta_{i,j} \bar{\phi}_{n-j}(z) \\ = \sum_{j=0}^n \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \overline{\phi_{n-i}(\zeta)} \phi_{n-j}(\zeta) \bar{\phi}_{n-j}(z) \\ = \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \overline{\phi_{n-i}(\zeta)} \sum_{j=0}^n \phi_{n-j}(\zeta) \overline{\phi_{n-j}(z)} \\ = \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \overline{\phi_{n-i}(\zeta)} \frac{\phi_n^*(\zeta) \overline{\phi_n^*(z)} - \zeta z \phi_n(\zeta) \overline{\phi_n(z)}}{1 - \zeta z}.$$

Writing the Kronecker delta in a similar way the whole expression becomes

$$0 = \int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \overline{\phi_{n-i}(\zeta)} \left\{ \dot{\phi}_n(\zeta) + \left(\frac{\dot{\kappa}_n}{\kappa_n} + \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} \right) \phi_n(\zeta) \right. \\ \left. - 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} \epsilon_n(z_j) \frac{\phi_n^*(\zeta) z_j^{-n} \phi_n(z_j) - \zeta z_j^{-1} \phi_n(\zeta) z_j^{-n} \phi_n^*(z_j)}{1 - \zeta z_j^{-1}} \right\}$$

for all $0 \leq i \leq n$ and (3.53) then follows. The second relation follows by an identical argument applied to

$$\int \frac{d\zeta}{2\pi i \zeta} w(\zeta) \phi_{n-i}(\zeta) \overline{\phi_n(\zeta)} = \delta_{i,0}.$$

The derivatives of the associated functions (3.55), (3.56) follow from differentiating the definitions (2.24), (2.25) and employing the first two results of the proposition along with the relation (2.30). \square

Corollary 3.2. *The t -derivatives of the reflection coefficients are*

$$(3.57) \quad \frac{\dot{r}_n}{r_n} = 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} \frac{\Omega_{n-1}(z_j) - V(z_j)}{V(z_j)}$$

$$(3.58) \quad \frac{\dot{\bar{r}}_n}{\bar{r}_n} = 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} \frac{\Omega_{n-1}^*(z_j) + V(z_j)}{V(z_j)}$$

Proof. An alternative form to (3.53) is

$$(3.59) \quad \dot{\phi}_n(\zeta) = - \left(\frac{\dot{\kappa}_n}{\kappa_n} + \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} \right) \phi_n(\zeta) + 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} \epsilon_n(z_j) \sum_{l=0}^n \bar{\phi}_{n-l}(z_j^{-1}) \phi_{n-l}(\zeta),$$

and by examining the coefficients of ζ^n, ζ^0 we deduce that

$$\frac{\dot{r}_n}{r_n} = 1/2 \frac{\kappa_{n-1}}{\phi_n(0)} \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{1-n} \epsilon_n(z_j) \phi_{n-1}(z_j).$$

Noting that the derivative term of (3.21) vanishes when $z = z_j$ and employing (2.28) we arrive at (3.57). The second equation, (3.58), follows by identical reasoning. \square

Sums of the bilinear residues over the singular points are related to deformation derivatives in the following way,

$$(3.60) \quad \begin{aligned} 2 \frac{\dot{\kappa}_n}{\kappa_n} &= - \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} + 1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n(z_j) \phi_n^*(z_j), \\ &= -1/2 \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n^*(z_j) \phi_n(z_j), \end{aligned}$$

$$(3.61) \quad \begin{aligned} \frac{\dot{\phi}_n(0)}{\phi_n(0)} + \frac{\dot{\kappa}_n}{\kappa_n} + \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} &= 1/2 \frac{\kappa_n}{\phi_n(0)} \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n(z_j) \phi_n(z_j), \\ &= \frac{\phi_{n+1}(0)}{\phi_n(0)} \sum_{j=1}^m \frac{\rho_j}{2V(z_j)} \dot{z}_j z_j^{-1} \Theta_n(z_j) \end{aligned}$$

$$(3.62) \quad \begin{aligned} \frac{\dot{\bar{\phi}}_n(0)}{\bar{\phi}_n(0)} + \frac{\dot{\kappa}_n}{\kappa_n} &= -1/2 \frac{\kappa_n}{\bar{\phi}_n(0)} \sum_{j=1}^m \rho_j \frac{\dot{z}_j}{z_j} z_j^{-n} \epsilon_n^*(z_j) \phi_n^*(z_j) \\ &= \frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)} \sum_{j=1}^m \frac{\rho_j}{2V(z_j)} \dot{z}_j \Theta_n^*(z_j). \end{aligned}$$

For the regular semi-classical weights we can also formulate the system of deformation derivatives as a 2×2 matrix differential equation and demonstrate that the system preserves the monodromy data with respect to the motion of the singularities $z_j(t)$.

Corollary 3.3. *The deformation derivatives for a system of regular semi-classical orthogonal polynomials and associated functions (3.53-3.56) are equivalent to the matrix differential equation*

$$(3.63) \quad \dot{Y}_n := B_n Y_n = \left\{ B_\infty - \sum_{j=1}^m \frac{\dot{z}_j}{z - z_j} A_{nj} \right\} Y_n.$$

where

$$(3.64) \quad B_\infty = \begin{pmatrix} \frac{\dot{\kappa}_n}{\kappa_n} & 0 \\ \frac{\kappa_n \dot{\phi}_n(0) + \dot{\kappa}_n \bar{\phi}(0)}{\kappa_n^2} & -\frac{\dot{\kappa}_n}{\kappa_n} \end{pmatrix}.$$

Proof. This follows from a partial fraction decomposition of the system (3.53-3.56) and using (3.60,3.62). \square

In the case of the pair (2.53), (3.63) compatibility implies the relation

$$(3.65) \quad \dot{M}_n = B_{n+1}M_n - M_nB_n,$$

however there are no new identities arising from this condition. Taking the 11-component of both sides of this equation we see that it is identically satisfied through the use of (2.46) and (3.62). Or if we take the 12-components then they are equal when use of made of (2.41) and (3.60,3.57). In a similar way we find both sides of the 21-components are identical when we employ (2.43) and (3.62). Finally the 22-components on both sides are the same after taking into account (2.48) and (3.60,3.62).

For the pair of linear differential relations (2.54), (3.63) compatibility leads us to the Schlesinger equations

$$(3.66) \quad \dot{A}_{nj} = [B_\infty, A_{nj}] + \sum_{k \neq j} \frac{\dot{z}_j - \dot{z}_k}{z_j - z_k} [A_{nk}, A_{nj}].$$

Again there is not anything essentially new here, that couldn't be derived from the system of deformation derivatives (3.53-3.56), but it is an efficient way to compute the deformation derivatives of bilinear products. Employing the explicit representations of our matrices A_k we find the following independent derivatives in component form

$$(3.67) \quad \begin{aligned} \frac{d}{dt} \frac{\rho_j}{2V(z_j)} \left[\Omega_n(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) \right] &= -\frac{\rho_j}{2V(z_j)} \frac{\phi_{n+1}(0)}{\kappa_n^3} \frac{d}{dt} (\kappa_n \bar{\phi}_n(0)) \Theta_n(z_j) \\ &\quad - \frac{\rho_j}{2V(z_j)} \frac{|\phi_{n+1}(0)|^2}{\kappa_n^2} \sum_{k \neq j} \frac{\dot{z}_j - \dot{z}_k}{z_j - z_k} \frac{\rho_k}{2V(z_k)} [z_k \Theta_n^*(z_k) \Theta_n(z_j) - z_j \Theta_n(z_k) \Theta_n^*(z_j)], \end{aligned}$$

$$(3.68) \quad \begin{aligned} \frac{d}{dt} \frac{\rho_j}{2V(z_j)} \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z_j) &= \frac{\rho_j}{V(z_j)} \frac{\phi_{n+1}(0)}{\kappa_n} \left\{ \frac{\dot{\kappa}_n}{\kappa_n} \Theta_n(z_j) + \sum_{k \neq j} \frac{\dot{z}_j - \dot{z}_k}{z_j - z_k} \frac{\rho_k}{2V(z_k)} \right. \\ &\quad \times \left. \left[\Theta_n(z_k) \left[\Omega_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) \right] - \Theta_n(z_j) \left[\Omega_n(z_k) - \frac{\kappa_{n+1}}{\kappa_n} z_k \Theta_n(z_k) \right] \right] \right\}, \end{aligned}$$

$$(3.69) \quad \begin{aligned} \frac{d}{dt} \frac{\rho_j}{2V(z_j)} \frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z_j \Theta_n^*(z_j) &= \frac{\rho_j}{V(z_j)} \frac{\bar{\phi}_{n+1}(0)}{\kappa_n} \left\{ -\frac{\dot{\kappa}_n}{\kappa_n} z_j \Theta_n^*(z_j) \right. \\ &\quad + \frac{1}{\kappa_n \bar{\phi}_{n+1}(0)} \frac{d}{dt} (\kappa_n \bar{\phi}_n(0)) [\Omega_n^*(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j)] - \sum_{k \neq j} \frac{\dot{z}_j - \dot{z}_k}{z_j - z_k} \frac{\rho_k}{2V(z_k)} \\ &\quad \times \left. \left[z_k \Theta_n^*(z_k) [\Omega_n^*(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j)] - z_j \Theta_n^*(z_j) [\Omega_n^*(z_k) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_k)] \right] \right\}. \end{aligned}$$

Remark 3.5. The fact that the deformation equations satisfy the Schlesinger system of partial differential equations should be of no great surprise as the isomonodromic properties of the regular semi-classical weights are quite transparent. In the neighbourhood of any isolated singularity $|z - z_j| < \delta$ the Carathéodory function can be decomposed

$$(3.70) \quad F(z) = f_j(z) + C_j w(z),$$

where $f_j(z)$ is the unique, holomorphic function in this neighbourhood and C_j a coefficient. The analytic continuation of Y_n around a closed loop enclosing the singularity is easily found and defines the monodromy matrix M_j ,

$$(3.71) \quad Y_n|_{z_j+\delta e^{2\pi i}} = Y_n|_{z_j+\delta} M_j, \quad M_j = \begin{pmatrix} 1 & C_j(1 - e^{-2\pi i \rho_j}) \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}.$$

From the definition (2.22) one can confirm that $C_j = 1/i \sin(\pi \rho_j)$, so that the M_j is quite naturally independent of the deformation variables z_j or t (ρ_j being constant).

4. THE SIMPLEST SEMI-CLASSICAL CLASS: P_{VI} SYSTEM

Here we consider the application of the general theory above to the simplest instance of the semi-classical weight, namely $m = 3$ singular points with two fixed at $z = 0, -1$ and the third a variable at $z = -1/t$.

Explicitly we consider the unitary group average (1.3) where $t = e^{i\phi}$, $\omega_1, \omega_2, \mu \in \mathbb{C}$ ($\omega, \bar{\omega} = \omega_1 \pm i\omega_2$) and $\xi \in \mathbb{C}$. In the initial formulation $t \in \mathbb{T}$ but will be analytically continued off the unit circle. The weight function

$$(4.1) \quad w(z) = t^{-\mu} z^{-\mu-\omega} (1+z)^{2\omega_1} (1+tz)^{2\mu} \begin{cases} 1 & \theta \notin (\pi - \phi, \pi) \\ 1 - \xi & \theta \in (\pi - \phi, \pi) \end{cases},$$

is known as a generalised Jacobi weight, with branch points at $z = 0, -1, -1/t, \infty$. When $t \in \mathbb{T}$, $\mu, \omega_1, \omega_2, \xi \in \mathbb{R}$ and $\xi < 1$ this weight is real and positive. The Toeplitz matrix is then hermitian and as a consequence \bar{r}_n is the complex conjugate of r_n , but generally this is not the case. The Toeplitz matrix elements can be evaluated in terms of the Gauss hypergeometric function, and there are several forms this can take which exhibit manifest analyticity at either of the special points $t = 0, 1, \infty$.

Lemma 4.1. *The Toeplitz matrix element w_n for the weight (4.1), under the restriction $\Re(\mu), \Re(\omega_1) > -1/2$, is given in terms of hypergeometric functions analytic at $t = 0, 1$*

$$(4.2) \quad \begin{aligned} t^\mu w_n &= \frac{\Gamma(2\omega_1 + 1)}{\Gamma(1 + n + \mu + \omega)\Gamma(1 - n - \mu + \bar{\omega})} {}_2F_1(-2\mu, -n - \mu - \omega; 1 - n - \mu + \bar{\omega}; t) \\ &+ \frac{\xi}{2\pi i} e^{\pm \pi i(n + \mu - \bar{\omega})} \frac{\Gamma(2\mu + 1)\Gamma(2\omega_1 + 1)}{\Gamma(2\mu + 2\omega_1 + 2)} t^{n + \mu - \bar{\omega}} (1 - t)^{2\mu + 2\omega_1 + 1} \\ &\quad \times {}_2F_1(2\mu + 1, 1 + n + \mu + \omega; 2\mu + 2\omega_1 + 2; 1 - t) \end{aligned}$$

where the \pm sign is taken accordingly as $\text{Im}(t) \geq 0$. This can also be written as

$$(4.3) \quad t^\mu w_n = \left\{ 1 + \xi \frac{e^{\pm \pi i(n+\mu-\bar{\omega})}}{2i \sin \pi(n+\mu-\bar{\omega})} \right\} \frac{\Gamma(2\omega_1+1)}{\Gamma(1+n+\mu+\omega)\Gamma(1-n-\mu+\bar{\omega})} \\ \times {}_2F_1(-2\mu, -n-\mu-\omega; 1-n-\mu+\bar{\omega}; t) \\ - \xi \frac{e^{\pm \pi i(n+\mu-\bar{\omega})}}{2i \sin \pi(n+\mu-\bar{\omega})} \frac{\Gamma(2\mu+1)}{\Gamma(1+n+\mu-\bar{\omega})\Gamma(1-n+\mu+\bar{\omega})} \\ \times t^{n+\mu-\bar{\omega}} (1-t)^{2\mu+2\omega_1+1} {}_2F_1(2\mu+1, 1+n+\mu+\omega; 1+n+\mu-\bar{\omega}; t).$$

Proof. This follows from the generalisation of the Euler integral for the Gauss hypergeometric function and consideration of the consistent phases for the branch cuts linking the singular points, see [33] pp. 91, section 17 "Verallgemeinerung der Eulersche Integrale". \square

Remark 4.1. The first of these forms was given in [20].

For the description of (4.1) in terms of the semi-classical form (3.2) we have

$$(4.4) \quad m = 3, \quad \{z_j\}_{j=1}^3 = \{0, -1, -1/t\}, \quad \{\rho_j\}_{j=1}^3 = \{-\mu - \omega, 2\omega_1, 2\mu\},$$

$$(4.5) \quad 2V(0) = -(\mu + \omega)t^{-1}, \quad W'(0) = t^{-1}, \quad V(-t^{-1}) = \mu \frac{1-t}{t^2}$$

and the coefficient functions are

$$(4.6) \quad \Theta_N(z) = \frac{\kappa_N}{\kappa_{N+1}} \left[(N+1+\mu+\bar{\omega})z - \frac{r_N}{r_{N+1}}(N+\mu+\omega)t^{-1} \right]$$

$$(4.7) \quad \Theta_N^*(z) = \frac{\kappa_N}{\kappa_{N+1}} \left[-\frac{\bar{r}_N}{\bar{r}_{N+1}}(N+\mu+\bar{\omega})z + (N+1+\mu+\omega)t^{-1} \right]$$

$$(4.8)$$

$$\Omega_N(z) = [1 + \tfrac{1}{2}(\mu + \bar{\omega})]z^2 \\ + \left\{ (N+2+\mu+\bar{\omega})(1-r_{N+1}\bar{r}_{N+1})\frac{r_{N+2}}{r_{N+1}} - \frac{l_{N+1}}{\kappa_{N+1}} + [1 + \tfrac{1}{2}(\mu + \bar{\omega})]\frac{1+t}{t} - \omega_1 - \frac{\mu}{t} \right\} z \\ - [N + \tfrac{1}{2}(\mu + \omega)]t^{-1}$$

$$(4.9)$$

$$\Omega_N^*(z) = -\tfrac{1}{2}(\mu + \bar{\omega})z^2 \\ + \left\{ \frac{l_{N+1}}{\kappa_{N+1}} - (N+\mu+\bar{\omega})(1-r_{N+1}\bar{r}_{N+1})\frac{\bar{r}_N}{\bar{r}_{N+1}} - \tfrac{1}{2}(\mu + \bar{\omega})\frac{1+t}{t} + \omega_1 + \frac{\mu}{t} \right\} z \\ + [N+1+\tfrac{1}{2}(\mu + \omega)]t^{-1}$$

Our objective is to show that the average (1.3) can be evaluated via recurrence relations for the reflection coefficients. We begin with some preliminary lemmas.

Lemma 4.2. *The reflection coefficients for the weight (4.1) satisfy the homogeneous second-order difference equation*

$$(4.10) \quad (N+1+\mu+\bar{\omega})tr_{N+1}\bar{r}_N - (N-1+\mu+\bar{\omega})tr_N\bar{r}_{N-1} \\ = (N+1+\mu+\omega)\bar{r}_{N+1}r_N - (N-1+\mu+\omega)\bar{r}_Nr_{N-1}.$$

Proof. The result of the above lemma, (4.10) can be found immediately from the general theory of Section 2, in many ways. By equating coefficients of z in the functional-difference equation (2.47) using (4.6, 4.7, 4.8, 4.9), all are trivially satisfied except for the z coefficient, which is precisely (4.10). Similarly starting with (2.48) and employing (4.6, 4.7, 4.9), one finds (4.10). Alternatively one could start with either (2.49) or (2.51) and arrive at the same result \square

Corollary 4.1. *The sub-leading coefficients l_N, \bar{l}_N satisfy the linear inhomogeneous equation*

$$(4.11) \quad (N + \mu + \bar{\omega})tl_N - (N + \mu + \omega)\bar{l}_N = N[\mu(t-1) + \bar{\omega} - \omega t]\kappa_N.$$

Proof. By substituting the general expression for the first difference of l_N, \bar{l}_N using (2.11) in (4.10) one finds that it can be summed exactly to yield

$$(4.12) \quad (N+1+\mu+\bar{\omega})t\frac{l_{N+1}}{\kappa_{N+1}} - (N+1+\mu+\omega)\frac{\bar{l}_{N+1}}{\kappa_{N+1}} \\ - (N+\mu+\bar{\omega})t\frac{l_N}{\kappa_N} + (N+\mu+\omega)\frac{\bar{l}_N}{\kappa_N} = \mu(t-1) + \bar{\omega} - \omega t.$$

This can be summed once more to yield the stated result. \square

Remark 4.2. One could alternatively proceed via the Freud approach [23] (see also [21]) and consider the integral

$$(4.13) \quad \int_{\mathbb{T}} \frac{dz}{2\pi iz} (1+z)(1+tz) \left[-\frac{\mu+\omega}{z} + \frac{2\omega_1}{1+z} + \frac{2\mu t}{1+tz} \right] w(z)\phi_N(z)\overline{\phi_N(z)}.$$

Here we recognise the logarithmic derivative of the weight function in the integrand

$$(4.14) \quad \frac{w'}{w} = -\frac{\mu+\omega}{z} + \frac{2\omega_1}{1+z} + \frac{2\mu t}{1+tz},$$

and by evaluating the integral in the two ways we find a linear equation for l_N , namely (4.12).

Lemma 4.3. *The sub-leading coefficients are related to the reflection coefficients by*

$$(4.15)$$

$$(4.16) \quad \begin{aligned} \bar{l}_N/\kappa_N + tl_N/\kappa_N - N(t+1) &= \frac{1-r_N\bar{r}_N}{r_N} [(N+1+\mu+\bar{\omega})tr_{N+1} + (N-1+\mu+\omega)r_{N-1}] \\ &= \frac{1-r_N\bar{r}_N}{\bar{r}_N} [(N+1+\mu+\omega)\bar{r}_{N+1} + (N-1+\mu+\bar{\omega})t\bar{r}_{N-1}] \end{aligned}$$

Proof. The first relation follows from a comparison of the coefficients of z for $\Omega_N(z)$ given the two distinct expansions, the first by (3.8) which reduces to (4.8) and the second by the specialisation of (3.9). The second relation follows from identical arguments applied to $\Omega_N^*(z)$ or by employing (4.10) in the first relation. \square

Remark 4.3. The first result appears in the Magnus derivation [35] for the generalised Jacobi weight, with $\theta_1 = \pi - \phi, \theta_2 = \pi, \alpha = \mu, \beta = \omega_1, \gamma = -\omega_2$. Then Eq. (14) of that work is precisely (4.15).

Remark 4.4. The Magnus relation (4.15) can also be found by employing the Freud method. In this one uses integration by parts on the integral

$$(4.17) \quad \int_{\mathbb{T}} \frac{dz}{2\pi iz} z^{-1}(1+z)(1+tz)w'(z)\phi_{N+1}(z)\overline{\phi_N(z)},$$

and in the term involving $\phi'_{N+1}(z)$ one employs (2.35) for the derivative and (4.6), (4.8) for the coefficient functions. Equating this expression to a direct evaluation of the integral then yields (4.15).

Lemma 4.4. *The sub-leading coefficient l_N can be expressed in terms of the reflection coefficients in the following ways*

$$(4.18) \quad 2t \frac{l_N}{\kappa_N} = (N+1+\mu+\bar{\omega})t \left(\frac{r_{N+1}}{r_N} - r_{N+1}\bar{r}_N \right) + (N-1+\mu+\omega) \frac{r_{N-1}}{r_N}$$

$$(4.19) \quad \begin{aligned} & - (N-1+\mu+\bar{\omega})r_N\bar{r}_{N-1} + (N+\mu-\omega)t + N - \mu + \bar{\omega} \\ & = (N+1+\mu+\omega) \frac{\bar{r}_{N+1}}{\bar{r}_N} + (N-1+\mu+\bar{\omega})t \left(\frac{\bar{r}_{N-1}}{\bar{r}_N} - r_N\bar{r}_{N-1} \right) \\ & - (N+1+\mu+\bar{\omega})tr_{N+1}\bar{r}_N + (N+\mu-\omega)t + N - \mu + \bar{\omega} \end{aligned}$$

as well as analogous expressions for \bar{l}_N .

Proof. The first expression follows from a comparison of the z^0 coefficients for $\Theta_N(z)$ evaluated using both (3.4) and (3.5). The second relation follows from applying the same reasoning to $\Theta_N^*(z)$. \square

We will refer to the order of a system of coupled difference equations with two variables r_n, \bar{r}_n say as q/p where $q \in \mathbb{Z}_{\geq 0}$ refers to the order of r_n and $p \in \mathbb{Z}_{\geq 0}$ refers to the order of \bar{r}_n .

Corollary 4.2. *The reflection coefficients of the OPS for the weight (4.1) satisfy the 2/2 order recurrence relations*

$$(4.20) \quad \begin{aligned} tr_N\bar{r}_{N-1} + r_{N-1}\bar{r}_N - t - 1 &= \frac{1 - r_N\bar{r}_N}{r_N} [(N+1+\mu+\bar{\omega})tr_{N+1} + (N-1+\mu+\omega)r_{N-1}] \\ &- \frac{1 - r_{N-1}\bar{r}_{N-1}}{\bar{r}_{N-1}} [(N+\mu+\omega)\bar{r}_N + (N-2+\mu+\bar{\omega})t\bar{r}_{N-2}] \end{aligned}$$

$$(4.21) \quad \begin{aligned} &= \frac{1 - r_N\bar{r}_N}{\bar{r}_N} [(N+1+\mu+\omega)\bar{r}_{N+1} + (N-1+\mu+\bar{\omega})t\bar{r}_{N-1}] \\ &- \frac{1 - r_{N-1}\bar{r}_{N-1}}{r_{N-1}} [(N+\mu+\bar{\omega})tr_N + (N-2+\mu+\omega)r_{N-2}] \end{aligned}$$

and those specifying the solution for (1.3) have the initial values

$$(4.22) \quad r_0 = \bar{r}_0 = 1, \quad r_1 = -w_{-1}/w_0, \quad \bar{r}_1 = -w_1/w_0,$$

where the Toeplitz matrix elements are given in (4.3).

Proof. Solving (4.15) for the combination of l_N, \bar{l}_N and differencing this, one arrives at (4.20). This however is of order 3/1 but by employing (4.10) we can reduce the order in r_N of the recurrence to second order. The other member of the pair (4.21) is found in an identical manner starting with (4.16). \square

Remark 4.5. The second-order difference (4.20) also follows immediately from equating the polynomials in z arising in the functional-difference (2.42), after employing (4.6,4.8). The other member of the pair, (4.21), follows from the functional-difference (2.44), after using (4.7,4.9).

Remark 4.6. In their most general example Adler and van Moerbeke also considered this weight. In terms of their variables we should set $P_1 = P_2 = 0, d_1 = t^{-1/2}, d_2 = t^{1/2}$, and without loss of generality $\gamma_1'' = \gamma_2' = 0$. For the other parameters $\gamma = \mu - \omega, \gamma_1' = 2\omega_1, \gamma_2'' = 2\mu$. There is a slight difference in the dependent variables due to the additional factor of t , so that we have the identification $x_N = (-1)^N t^{N/2} r_N, y_N = (-1)^N t^{-N/2} \bar{r}_N$ and $v_N = 1 - r_N \bar{r}_N$. Generalising their working one finds that their Eq. (0.0.14) implies

$$(4.23) \quad - (N+1+\mu+\bar{\omega})x_{N+1}y_N + (N+1+\mu+\omega)x_N y_{N+1} \\ + (N-1+\mu+\bar{\omega})x_N y_{N-1} - (N-1+\mu+\omega)x_{N-1}y_N = 0.$$

Now by transforming to our r_N, \bar{r}_N and employing (2.11) one finds this is precisely (4.10), which we showed is solved by (4.11). Their inhomogeneous Eq. (0.0.15) now takes the form

$$(4.24) \quad -v_N [(N+1+\mu+\bar{\omega})x_{N+1}y_{N-1} + N+\mu+\omega] \\ + v_{N-1} [(N-2+\mu+\bar{\omega})x_N y_{N-2} + N-1+\mu+\omega] \\ + x_N y_{N-1} (x_N y_{N-1} + t^{1/2} + t^{-1/2}) \\ = -v_1 [(2+\mu+\bar{\omega})x_2 + 1+\mu+\omega] + x_1 (x_1 + t^{1/2} + t^{-1/2}).$$

Upon recasting this into our variables and manipulating, it then becomes

$$(4.25) \quad tr_N \bar{r}_{N-1} + \bar{r}_N r_{N-1} - t - 1 - \frac{1 - r_N \bar{r}_N}{r_N} [(N+1+\mu+\bar{\omega})tr_{N+1} + (N-1+\mu+\omega)r_{N-1}] \\ + \frac{1 - r_{N-1} \bar{r}_{N-1}}{\bar{r}_{N-1}} [(N+\mu+\omega)\bar{r}_N + (N-2+\mu+\bar{\omega})t\bar{r}_{N-2}] \\ = \frac{1 - (1 - r_1 \bar{r}_1) [(2+\mu+\bar{\omega})tr_2 + 1+\mu+\omega] + r_1 (tr_1 - t - 1)}{r_N \bar{r}_{N-1}}.$$

However using the identity

$${}_2F_1(a, b; c; x) = [c + (1+b-a)x] {}_2F_1(a, b+1; c+1; x) - \frac{b+1}{c+1} (1+c-a) {}_2F_1(a, b+2; c+2; x)$$

we note that the right-hand side is identically zero for the initial conditions (4.22) and the recurrence is not genuinely inhomogeneous, thus yielding our first relation above, (4.20).

We seek recurrences for r_N, \bar{r}_N which are of the form of the discrete Painlevé system (1.8), (1.9). For this purpose a number of distinct forms of the former will be presented.

Proposition 4.1. *The reflection coefficients satisfy a system of a 2/0 order recurrence relation*

$$\begin{aligned}
 (4.26) \quad & \left\{ (1 - r_N \bar{r}_N) [(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1} - (N+\mu+\omega)(N-1+\mu+\omega)r_{N-1}] \right. \\
 & \quad \left. + N(N+2\omega_1)(t-1)r_N \right\} \\
 & \times \left\{ (1 - r_N \bar{r}_N) [(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1} - (N+\mu+\omega)(N-1+\mu+\omega)r_{N-1}] \right. \\
 & \quad \left. + (N+2\mu)(N+2\mu+2\omega_1)(t-1)r_N \right\} \\
 & = -(2N+2\mu+2\omega_1)^2 t(1 - r_N \bar{r}_N) \\
 & \times [(N+1+\mu+\bar{\omega})r_{N+1} + (N+\mu+\omega)r_N] [(N+\mu+\bar{\omega})r_N + (N-1+\mu+\omega)r_{N-1}]
 \end{aligned}$$

and a 0/2 order recurrence relation which is just (4.26) with the replacements $\omega \leftrightarrow \bar{\omega}$ and $t^{\pm 1/2} r_j \mapsto t^{\mp 1/2} \bar{r}_j$

Proof. Consider first the specialisation of (3.44) to our weight at hand at the singular point $z = -1$, and we have

$$\begin{aligned}
 (4.27) \quad & \left\{ \frac{l_N}{\kappa_N} - Nt^{-1} - (N+1+\mu+\bar{\omega}) \frac{\kappa_{N-1}^2}{\kappa_N^2} \frac{r_{N+1}}{r_N} + \omega_1(1-t^{-1}) \right\}^2 \\
 & + \frac{\kappa_{N-1}^2}{\kappa_N^2} \left[N + \mu + \bar{\omega} + \frac{(N-1+\mu+\omega)}{t} \frac{r_{N-1}}{r_N} \right] \left[\frac{(N+\mu+\omega)}{t} + (N+1+\mu+\bar{\omega}) \frac{r_{N+1}}{r_N} \right] \\
 & = \omega_1^2 \left(\frac{t-1}{t} \right)^2,
 \end{aligned}$$

by using (4.6,4.8). Similarly (3.44) evaluated at $z = -1/t$ yields

$$\begin{aligned}
 (4.28) \quad & \left\{ \frac{l_N}{\kappa_N} - N - (N+1+\mu+\bar{\omega}) \frac{\kappa_{N-1}^2}{\kappa_N^2} \frac{r_{N+1}}{r_N} + \mu(t^{-1}-1) \right\}^2 \\
 & + \frac{\kappa_{N-1}^2}{\kappa_N^2} t^{-1} \left[N + \mu + \bar{\omega} + (N-1+\mu+\omega) \frac{r_{N-1}}{r_N} \right] \left[N + \mu + \omega + (N+1+\mu+\bar{\omega}) \frac{r_{N+1}}{r_N} \right] \\
 & = \mu^2 \left(\frac{t-1}{t} \right)^2.
 \end{aligned}$$

The first relation follow by eliminating l_N between (4.27) and (4.28), whereas the second follows from an identical analysis to that employed in the proof of Proposition 4.4 but starting with the bilinear identity (3.45). \square

Proposition 4.2. *The reflection coefficients also satisfy a system of 1/1 order recurrence relations the first of which is*

$$\begin{aligned}
 (4.29) \quad & \left\{ - (N + \mu + \omega)(1 - r_N \bar{r}_N) t [(N + 1 + \mu + \bar{\omega}) r_{N+1} \bar{r}_N + (N - 1 + \mu + \bar{\omega}) r_N \bar{r}_{N-1}] \right. \\
 & + 2(N + \mu + \omega)^2 r_N^2 \bar{r}_N^2 - (N + \mu + \omega)^2 (t + 1) r_N \bar{r}_N - 2(N + \mu + \omega) \bar{\omega} (t - 1) r_N \bar{r}_N \\
 & \left. + (\mu - \bar{\omega})(\mu + \bar{\omega})(t - 1) \right\} \\
 & \times \left\{ - (N + \mu + \omega)(1 - r_N \bar{r}_N) t [(N + 1 + \mu + \bar{\omega}) r_{N+1} \bar{r}_N + (N - 1 + \mu + \bar{\omega}) r_N \bar{r}_{N-1}] \right. \\
 & + 2(N + \mu + \omega)^2 r_N^2 \bar{r}_N^2 - (N + \mu + \omega)^2 (t + 1) r_N \bar{r}_N + 2(N + \mu + \omega) \omega (t - 1) r_N \bar{r}_N \\
 & \left. + (\mu - \omega)(\mu + \omega)(t - 1) \right\} = - [2(N + \mu + \omega) r_N \bar{r}_N + \bar{\omega} - \omega]^2 (1 - r_N \bar{r}_N) \\
 & \times [(N + 1 + \mu + \bar{\omega}) t r_{N+1} + (N + \mu + \omega) r_N] [(N + \mu + \omega) \bar{r}_N + (N - 1 + \mu + \bar{\omega}) t \bar{r}_{N-1}]
 \end{aligned}$$

and the second is obtained from (4.29) with the replacements $\omega \leftrightarrow \bar{\omega}$ and $t^{\pm 1/2} r_j \leftrightarrow t^{\mp 1/2} \bar{r}_j$.

Proof. The specialisation of (3.46) to the weight (4.1) evaluated at the singular point $z = -1$ is

$$\begin{aligned}
 (4.30) \quad & \left\{ \frac{l_N}{\kappa_N} - N t^{-1} + (N + \mu + \omega) t^{-1} \frac{\kappa_{N-1}^2}{\kappa_N^2} + \omega_1 (1 - t^{-1}) \right\}^2 \\
 & + \frac{\kappa_{N-1}^2}{\kappa_N^2} [(N + 1 + \mu + \bar{\omega}) r_{N+1} + (N + \mu + \omega) t^{-1} r_N] \\
 & \times [(N - 1 + \mu + \bar{\omega}) \bar{r}_{N-1} + (N + \mu + \omega) t^{-1} \bar{r}_N] = \omega_1^2 \left(\frac{t-1}{t} \right)^2,
 \end{aligned}$$

by using (4.6, 4.8). Similarly (3.46) evaluated at $z = -1/t$ yields

$$\begin{aligned}
 (4.31) \quad & \left\{ \frac{l_N}{\kappa_N} - N + (N + \mu + \omega) \frac{\kappa_{N-1}^2}{\kappa_N^2} + \mu (t^{-1} - 1) \right\}^2 \\
 & + \frac{\kappa_{N-1}^2}{\kappa_N^2} [(N + 1 + \mu + \bar{\omega}) r_{N+1} + (N + \mu + \omega) r_N] \\
 & \times [(N - 1 + \mu + \bar{\omega}) \bar{r}_{N-1} + (N + \mu + \omega) \bar{r}_N] = \mu^2 \left(\frac{t-1}{t} \right)^2.
 \end{aligned}$$

Again eliminating l_N between these two equations yields the recurrence relation (4.29). The second follows in the same way starting with (3.47). \square

Proposition 4.3. *The reflection coefficients satisfy an alternative system of 1/1 order recurrence relations the first of which is*

$$\begin{aligned}
 (4.32) \quad & \left[(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1}\bar{r}_N \right. \\
 & \quad \left. - (N+1+\mu+\omega)(N+\mu+\omega)\bar{r}_{N+1}r_N + (\bar{\omega}-\mu)(\bar{\omega}+\mu)(t-1) \right] \\
 & \quad \times \left[(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1}\bar{r}_N \right. \\
 & \quad \left. - (N+1+\mu+\omega)(N+\mu+\omega)\bar{r}_{N+1}r_N + (\omega-\mu)(\omega+\mu)(t-1) \right] \\
 & \quad = (\bar{\omega}-\omega)^2 \\
 & \quad \times [(N+1+\mu+\bar{\omega})tr_{N+1} + (N+\mu+\omega)r_N] [(N+1+\mu+\omega)\bar{r}_{N+1} + (N+\mu+\bar{\omega})t\bar{r}_N]
 \end{aligned}$$

and the second is again obtained from (4.32) with the replacements $\omega \leftrightarrow \bar{\omega}$ and $t^{\pm 1/2}r_j \leftrightarrow t^{\mp 1/2}\bar{r}_j$.

Proof. The specialisation of (3.48) to the weight (4.1) evaluated at the singular point $z = -1$ is

$$\begin{aligned}
 (4.33) \quad & \left\{ \frac{\bar{l}_{N+1}}{\kappa_{N+1}} + (N+\mu+\omega)\bar{r}_{N+1}r_N + \omega_1 + (\mu - i\omega_2)t \right\}^2 \\
 & = [(N+1+\mu+\bar{\omega})tr_{N+1} + (N+\mu+\omega)r_N] \\
 & \quad \times [(N+1+\mu+\omega)\bar{r}_{N+1} + (N+\mu+\bar{\omega})t\bar{r}_N] + \omega_1^2(t-1)^2,
 \end{aligned}$$

by using (4.6,4.8). Similarly (3.48) evaluated at $z = -1/t$ yields

$$\begin{aligned}
 (4.34) \quad & \left\{ \frac{\bar{l}_{N+1}}{\kappa_{N+1}} + (N+\mu+\omega)\bar{r}_{N+1}r_N + \bar{\omega} + \mu t \right\}^2 \\
 & = [(N+1+\mu+\bar{\omega})r_{N+1} + (N+\mu+\omega)r_N] \\
 & \quad \times [(N+1+\mu+\omega)\bar{r}_{N+1} + (N+\mu+\bar{\omega})\bar{r}_N] + \mu^2(t-1)^2,
 \end{aligned}$$

Again eliminating \bar{l}_{N+1} between these two equations yields the recurrence relation (4.32). The second follows in the same way starting with (3.49). \square

Remark 4.7. Note that the recurrence system (4.26) and its partner is quadratic in r_{N+1} , r_{N-1} and \bar{r}_{N+1} , \bar{r}_{N-1} , the system (4.29) and its partner is also quadratic in r_{N+1} , \bar{r}_{N-1} and \bar{r}_{N+1} , r_{N-1} , and likewise (4.32) is quadratic in r_{N+1} , \bar{r}_{N+1} . This renders them less useful in practical iterations than the higher order systems that are linear in the highest difference. By raising the order of one of the variables by one we can obtain a recurrence linear in the highest difference.

Corollary 4.3. *The reflection coefficients satisfy a system of a 2/1 order recurrence relation*

$$\begin{aligned}
 (4.35) \quad & (N+1+\mu+\bar{\omega})(\bar{\omega}-\omega)t(1-r_N\bar{r}_N)r_{N+1} \\
 & + (N-1+\mu+\omega)[2(N+\mu+\omega)r_N\bar{r}_N + \bar{\omega}-\omega]r_{N-1} \\
 & - (N-1+\mu+\bar{\omega})(2N+2\mu+2\omega_1)tr_N^2\bar{r}_{N-1} \\
 & + [(\bar{\omega}-\omega)N(t+1) - (2\mu+2\omega_1)[\mu(1-t) + \omega t - \bar{\omega}]]r_n = 0
 \end{aligned}$$

and a $1/2$ order recurrence relation which is again obtained from (4.35) with the replacements $\omega \leftrightarrow \bar{\omega}$ and $t^{\pm 1/2} r_j \leftrightarrow t^{\mp 1/2} \bar{r}_j$.

Proof. The solutions for the sub-leading coefficient l_N, \bar{l}_N that arise from the simultaneous solution of (4.30,4.31) and (4.33,4.34) respectively are given by

$$(4.36) \quad t \frac{l_N}{\kappa_N} = \left\{ (N + \mu + \omega) t (1 - r_N \bar{r}_N) [(N + 1 + \mu + \bar{\omega}) r_{N+1} \bar{r}_N + (N - 1 + \mu + \bar{\omega}) r_N \bar{r}_{N-1}] \right. \\ \left. + (N + \mu + \omega) [N(t + 1) - \mu(1 - t) - \omega t + \bar{\omega}] r_N \bar{r}_N + (\omega + \mu) [\mu(1 - t) + \omega t - \bar{\omega}] \right\} \\ \div [2(N + \mu + \omega) r_N \bar{r}_N + \bar{\omega} - \omega]$$

$$(4.37) \quad t \frac{l_N}{\kappa_N} = \left\{ (N + \mu + \omega) [(N - 1 + \mu + \bar{\omega}) t r_N \bar{r}_{N-1} - (N - 1 + \mu + \omega) \bar{r}_N r_{N-1}] \right. \\ \left. + (\omega + \mu) [\mu(1 - t) + \omega t - \bar{\omega}] \right\} / (\bar{\omega} - \omega), \quad \text{if } \bar{\omega} \neq \omega,$$

and the corresponding expression for \bar{l}_N/κ_N under the above replacements. Equating these two forms then leads to (4.35). \square

The systems of recurrences that we have found are in fact equivalent to the discrete Painlevé equation associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$ and we give our first demonstration of this fact here.

Proposition 4.4. *The N -recurrence for the reflection coefficients of the orthogonal polynomial system with the weight (4.1) is governed by either of two systems of coupled first order discrete Painlevé equations (1.8), (1.9). This first is*

$$(4.38) \quad g_{N+1} g_N = t \frac{(f_N + N)(f_N + N + 2\mu)}{f_N(f_N - 2\omega_1)}$$

$$(4.39) \quad f_N + f_{N-1} = 2\omega_1 + \frac{N - 1 + \mu + \omega}{g_N - 1} + \frac{(N + \mu + \bar{\omega})t}{g_N - t},$$

subject to the initial conditions

$$(4.40) \quad g_1 = t \frac{\mu + \omega + (1 + \mu + \bar{\omega})r_1}{\mu + \omega + (1 + \mu + \bar{\omega})tr_1}, \quad f_0 = 0.$$

The transformations relating these variables to the reflection coefficients are given by

$$(4.41) \quad g_N = t \frac{N - 1 + \mu + \omega + (N + \mu + \bar{\omega}) \frac{r_N}{r_{N-1}}}{N - 1 + \mu + \omega + (N + \mu + \bar{\omega}) t \frac{r_N}{r_{N-1}}},$$

$$(4.42) \quad f_N = \frac{1}{1 - t} \left[t \frac{l_N}{\kappa_N} - N - (N + 1 + \mu + \bar{\omega})(1 - r_N \bar{r}_N) t \frac{r_{N+1}}{r_N} \right].$$

The second system is

$$(4.43) \quad \bar{g}_{N+1} \bar{g}_N = t^{-1} \frac{(\bar{f}_N + N)(\bar{f}_N + N + 2\omega_1)}{\bar{f}_N(\bar{f}_N - 2\mu)}$$

$$(4.44) \quad \bar{f}_N + \bar{f}_{N-1} = 2\mu + \frac{N + \mu + \omega}{\bar{g}_N - 1} + \frac{(N - 1 + \mu + \bar{\omega})t^{-1}}{\bar{g}_N - t^{-1}},$$

subject to the initial conditions

$$(4.45) \quad \bar{g}_1 = \frac{\mu + \bar{\omega} + (1 + \mu + \omega)t^{-1}\bar{r}_1}{\mu + \bar{\omega} + (1 + \mu + \omega)\bar{r}_1}, \quad \bar{f}_0 = 0.$$

The transformations relating these variables to the reflection coefficients are given by

$$(4.46) \quad \bar{g}_N = \frac{N - 1 + \mu + \bar{\omega} + (N + \mu + \omega)t^{-1}\frac{\bar{r}_N}{\bar{r}_{N-1}}}{N - 1 + \mu + \bar{\omega} + (N + \mu + \omega)\frac{\bar{r}_N}{\bar{r}_{N-1}}},$$

$$(4.47) \quad \bar{f}_N = \frac{1}{1 - t} \left[-t \frac{l_N}{\kappa_N} + Nt + (N - 1 + \mu + \bar{\omega})(1 - r_N \bar{r}_N)t \frac{\bar{r}_{N-1}}{\bar{r}_N} \right].$$

Proof. Consolidating each of (4.27) and (4.28) into two terms and taking their ratio then leads to (4.38) after utilising the definitions (4.41, 4.42). The second member of the recurrence system (4.39) follows from the relation

$$(4.48) \quad \frac{l_{N+1}}{\kappa_{N+1}} + \frac{l_N}{\kappa_N} \\ = (N + 2 + \mu + \bar{\omega})(1 - r_{N+1}\bar{r}_{N+1})\frac{r_{N+2}}{r_{N+1}} + (N + \mu + \omega)t^{-1}\frac{r_N}{r_{N+1}} - (N + 1 + \mu + \bar{\omega})r_{N+1}\bar{r}_N \\ - 2\omega_1 - 2\mu t^{-1} + (N + 1 + \mu + \bar{\omega})(1 + t^{-1}),$$

which results from a combination of (4.18) and (2.11), and the definition (4.42). All the results for the second system follow by applying identical reasoning starting with (3.45). \square

Remark 4.8. Generalised hypergeometric function evaluations were given in [22] in the special case $\xi = 0$. In terms of our unitary group average one such evaluation reads

$$(4.49) \quad \left\langle \prod_{l=1}^N z_l^{-\mu-\omega} (1 + z_l)^{2\omega_1} (1 + tz_l)^{2\mu} \right\rangle_{U(N)} \\ = \prod_{j=0}^{N-1} \frac{j! \Gamma(2\omega_1 + j + 1)}{\Gamma(1 + \mu + \omega + j) \Gamma(1 - \mu + \bar{\omega} + j)} \\ \times {}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \dots, t_N) \Big|_{t_1 = \dots = t_N = t},$$

subject to $\Re(\omega_1) > -1/2$ and $|t| < 1$. Similarly, for the reflection coefficients we have

$$(4.50) \quad r_N = (-1)^N \frac{(\mu + \omega)_N}{(1 - \mu + \bar{\omega})_N} \frac{{}_2F_1^{(1)}(-2\mu, 1 - \mu - \omega; N + 1 - \mu + \bar{\omega}; t_1, \dots, t_N)}{{}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \dots, t_N)} \Big|_{t_1 = \dots = t_N = t},$$

$$(4.51) \quad \bar{r}_N = (-1)^N \frac{(-\mu + \bar{\omega})_N}{(1 + \mu + \omega)_N} \frac{{}_2F_1^{(1)}(-2\mu, -1 - \mu - \omega; N - 1 - \mu + \bar{\omega}; t_1, \dots, t_N)}{{}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \dots, t_N)} \Big|_{t_1 = \dots = t_N = t}.$$

The analog of the Euler identity for this function is

$$(4.52) \quad {}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \dots, t_N) \Big|_{t_1 = \dots = t_N = 1} = \prod_{j=1}^N \frac{\Gamma(j + 2\mu + 2\omega_1) \Gamma(j - \mu + \bar{\omega})}{\Gamma(j + 2\omega_1) \Gamma(j + \mu + \bar{\omega})},$$

when $\Re(\mu + \omega_1) > -1/2$, $\Re(-\mu + \bar{\omega}) > -1$, thus implying Gamma function evaluations in the special case $t = 1$.

Remark 4.9. The degeneration of the P_{VI} system to the P_V system is facilitated by the replacements $\omega + \mu \mapsto \nu, \bar{\omega} - \mu \mapsto \mu, t \mapsto t/2\mu$ and then taking the limit $\mu \rightarrow \infty$. The coefficients of the orthogonal polynomials r_N, l_N remain of $O(1)$ in this limit. Then we see the explicit degeneration of the following equations - (4.11) \rightarrow Eq. (4.23)[21], the recurrence relations (4.29) \rightarrow Eq. (4.60)[21], (4.35) \rightarrow Eq. (4.9)[21] and its conjugate to Eq. (4.10)[21] modulo the identity Eq. (4.5)[21], and the hypergeometric functions (4.49) \rightarrow Eq. (4.24)[21], (4.50) \rightarrow Eq. (4.26)[21], and (4.51) \rightarrow Eq. (4.27)[21].

Remark 4.10. Two simple cases exist for the special values of the argument $t = 0, 1$. In the first case, $t = 0$, we have

$$(4.53) \quad r_N = (-1)^N \frac{(\mu + \omega)_N}{(1 - \mu + \bar{\omega})_N}, \quad \bar{r}_N = (-1)^N \frac{(-\mu + \bar{\omega})_N}{(1 + \mu + \omega)_N},$$

$$(4.54) \quad l_N = -\frac{(\mu + \omega)N}{(N - \mu + \bar{\omega})},$$

whereas for $t = 1$ (and ξ is irrelevant) we have

$$(4.55) \quad r_N = (-1)^N \frac{(\mu + \omega)_N}{(1 + \mu + \bar{\omega})_N}, \quad \bar{r}_N = (-1)^N \frac{(\mu + \bar{\omega})_N}{(1 + \mu + \omega)_N},$$

$$(4.56) \quad l_N = -\frac{(\mu + \omega)N}{(N + \mu + \bar{\omega})}.$$

There is a specialisation of the generalised Jacobi weights leading to a formulation of the orthogonal polynomial system in terms of real variables, and a simple phase factor appearing in the reflection coefficients. This occurs when $\mu, \omega_1 \in \mathbb{R}$, $\omega_2 = 0$ and $|t| = 1$ and is a special case of hermitian Toeplitz matrix elements. In such a situation \bar{r}_n is no longer independent of r_n (it is the complex conjugate of r_n) and the coupled systems of difference equations reduce to a single equation.

Corollary 4.4. *When $\omega_2 = 0$, $t \in \mathbb{T}$, $\mu, \omega_1 \in \mathbb{R}$ with $\mu + \omega_1 \notin \mathbb{Z}_{<0}$ and $\bar{r}_1 = tr_1$ then the reflection coefficients are products of a real coefficient $x_n \in \mathbb{R}$ and a phase factor so that $r_n = t^{-n/2}x_n, \bar{r}_n = t^{n/2}x_n$.*

Proof. Setting $\omega_2 = 0$ in (4.10) we note this can be rearranged as

$$(4.57) \quad (n+1+\mu+\omega) \left[t \frac{r_{n+1}}{r_n} - \frac{\bar{r}_{n+1}}{\bar{r}_n} \right] + (n-1+\mu+\omega) \left[\frac{r_{n-1}}{r_n} - t \frac{\bar{r}_{n-1}}{\bar{r}_n} \right] = 0.$$

Given that $\bar{r}_n = t^n r_n, \bar{r}_{n-1} = t^{n-1} r_{n-1}$ we use the above equality to show

$$(4.58) \quad \bar{r}_{n+1} = t^{n+1} r_{n+1},$$

and by induction on n the statement $\bar{r}_n = t^n r_n$ must be true $n \geq 0$ as it holds for $n = 0, 1$. The corollary then follows. \square

5. THE τ -FUNCTION THEORY FOR P_{VI}

In the Okamoto theory for P_{VI} the Hamiltonian function which governs the evolution of $\{q, p; H, t\}$ through the system (1.7) is

$$(5.1) \quad K := t(t-1)H \\ = q(q-1)(q-t)p^2 - [\alpha_4(q-1)(q-t) + \alpha_3q(q-t) + (\alpha_0-1)q(q-1)]p \\ + \alpha_2(\alpha_1 + \alpha_2)(q-t),$$

with parameters $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$ subject to the constraint $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$. In the introduction it was remarked that the unitary group average (1.3) was shown to be a τ -function for the sixth Painlevé system [22] and this can be achieved via two distinct methods. In the first method [20] the connection with the P_{VI} τ -function was established for an average with respect to the Cauchy unitary ensemble (see Eqs. (1.12, 1.19, 3.20, 3.28, 3.30) of [22]) and using the stereographic projection this average was related to (1.3) with the parameters

$$(5.2) \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (N+1+2\omega_1, N+2\mu, -N, -\mu-\omega, -\mu-\bar{\omega}).$$

The appropriate sequence of the Hamiltonian variables $\{q_n, p_n, H_n, \tau_n\}_{n=0,1,\dots}$ in which N is only incremented is generated by a shift operator $L_{01}^{-1} = r_1 s_0 s_1 s_2 s_3 s_4 s_2$ in terms of the reflection operators and Dynkin diagram automorphisms of the extended affine Weyl group $W_a(D_4^{(1)})$ (see [22]). It has the action $L_{01}^{-1} : \alpha_0 \mapsto \alpha_0 + 1, \alpha_1 \mapsto \alpha_1 + 1, \alpha_2 \mapsto \alpha_2 - 1$. For such a sequence we have the following result.

Lemma 5.1 ([20],[22]). *The sequence of auxiliary variables $\{g_n, f_n\}_{n=0,1,\dots}$ defined by*

$$(5.3) \quad g_n := \frac{q_n}{q_n - 1},$$

$$(5.4) \quad f_n := q_n(q_n - 1)p_n + (1 - \alpha_2 - \alpha_4)(q_n - 1) - \alpha_3 q_n - \alpha_0 \frac{q_n(q_n - 1)}{q_n - t}$$

generated by the shift operator L_{01}^{-1} satisfies the discrete Painlevé equations associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$

$$(5.5) \quad g_{n+1}g_n = \frac{t}{t-1} \frac{(f_n + 1 - \alpha_2)(f_n + 1 - \alpha_2 - \alpha_4)}{f_n(f_n + \alpha_3)}$$

$$(5.6) \quad f_n + f_{n-1} = -\alpha_3 + \frac{\alpha_1}{g_n - 1} + \frac{\alpha_0 t}{t(g_n - 1) - g_n}$$

Applying this result to the average (1.3) implies a recurrence scheme to compute the latter.

Proposition 5.1 ([20]). *Let $\{g_N\}_{N=0,1,\dots}, \{f_N\}_{N=0,1,\dots}$ satisfy the discrete Painlevé coupled difference equations associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$*

$$(5.7) \quad g_{N+1}g_N = \frac{t}{t-1} \frac{(f_N + N + 1)(f_N + N + 1 + \mu + \bar{\omega})}{f_N(f_N - \mu - \omega)}$$

$$(5.8) \quad f_N + f_{N-1} = \mu + \omega + \frac{N + 2\mu}{g_N - 1} + \frac{(N + 1 + 2\omega_1)t}{t(g_N - 1) - g_N},$$

where $t = 1/(1 - e^{i\phi})$ subject to the initial conditions

$$g_0 = \frac{q_0}{q_0 - 1}, \quad f_0 = (1 + \mu + \bar{\omega})(q_0 - 1) + (\mu + \omega)q_0 - (2\omega_1 + 1) \frac{q_0(q_0 - 1)}{q_0 - t}$$

with

$$(5.9) \quad q_0 = \frac{1}{2} \left(1 + \frac{i}{\mu} \frac{d}{d\phi} \log e^{i\mu\phi} T_1(e^{i\phi}) \right).$$

Define $\{q_N, p_N\}_{N=0,1,\dots}$ by

$$(5.10)$$

$$q_N = \frac{g_N}{g_N - 1},$$

$$(5.11)$$

$$p_N = \frac{(g_N - 1)^2}{g_N} f_N - (N + 1 + \mu + \bar{\omega}) \frac{g_N - 1}{g_N} - (\mu + \omega)(g_N - 1) + (N + 1 + 2\omega_1) \frac{g_N - 1}{t + (1 - t)g_N}.$$

Then with $T_0(e^{i\phi}) = 1$ and $T_1(e^{i\phi}) = w_0(e^{i\phi})$ as given by (4.2, 4.3), $\{T_N\}_{N=2,3,\dots}$ is specified by the recurrence

$$(5.12) \quad - (N + \mu + \omega)(N + \mu + \bar{\omega}) \frac{T_{N+1}T_{N-1}}{T_N^2} = q_N(q_N - 1)p_N^2 + (2\mu + 2\omega_1)q_N p_N - (\mu + \bar{\omega})p_N - N(N + 2\mu + 2\omega_1).$$

In the second method the connection with the P_{VI} τ -function was established for an average with respect to the Jacobi unitary ensemble (see Eqs. (1.21, 3.7, 3.28, 3.31, 3.32) of [22]) and using the projection $(-1, 1) \rightarrow \mathbb{T}$ under the condition $\xi = 0$ this average was related to (1.3) with the parameters

$$(5.13) \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1 - \mu - \omega, N + 2\mu, -N, -\mu - \bar{\omega}, N + 2\omega_1).$$

Sequences of the Hamiltonian variables $\{q_n, p_n, H_n, \tau_n\}_{n=0,1,\dots}$ are now generated by the shift operator $L_{14}^{-1} = r_3 s_1 s_4 s_2 s_0 s_3 s_2$. It has the action $L_{14}^{-1} : \alpha_1 \mapsto \alpha_1 + 1, \alpha_2 \mapsto \alpha_2 - 1, \alpha_4 \mapsto \alpha_4 + 1$. Using the methods of [22] we have the following result.

Lemma 5.2. *The sequence of auxiliary variables $\{g_n, f_n\}_{n=0,1,\dots}$ defined by*

$$(5.14) \quad g_n := \frac{q_n - t}{q_n - 1},$$

$$(5.15) \quad f_n := \frac{1}{1 - t} \left[(q_n - t)(q_n - 1)p_n + (1 - \alpha_0 - \alpha_2)(q_n - 1) - \alpha_3(q_n - t) - \alpha_4 \frac{(q_n - t)(q_n - 1)}{q_n} \right]$$

generated by the shift operator L_{14}^{-1} satisfies the discrete Painlevé equations associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$

$$(5.16) \quad g_{n+1}g_n = t \frac{(f_n + 1 - \alpha_2)(f_n + 1 - \alpha_0 - \alpha_2)}{f_n(f_n + \alpha_3)}$$

$$(5.17) \quad f_n + f_{n-1} = -\alpha_3 + \frac{\alpha_1}{g_n - 1} + \frac{\alpha_4 t}{g_n - t}$$

Proof. Using the action of the fundamental reflections and Dynkin diagram automorphisms given in Table 1 of [22] we compute the action of L_{14}^{-1} on q and write it in the following way,

$$\begin{aligned} \frac{(q-t)(\hat{q}-t)}{(q-1)(\hat{q}-1)} &= t[q(q-1)(q-t)p + (\alpha_1 + \alpha_2)q^2 - ((\alpha_0 + \alpha_1 + \alpha_2)t - \alpha_0 - \alpha_4)q - \alpha_4t] \\ &\quad \times [q(q-1)(q-t)p + (\alpha_1 + \alpha_2)q^2 - ((\alpha_1 + \alpha_2)t - \alpha_4)q - \alpha_4t] \\ &\quad \div [q(q-1)(q-t)p + (\alpha_1 + \alpha_2)q^2 - (-\alpha_4t + \alpha_1 + \alpha_2)q - \alpha_4t] \\ &\quad \div [q(q-1)(q-t)p + (\alpha_1 + \alpha_2)q^2 - (-\alpha_3 + \alpha_4)t + \alpha_1 + \alpha_2 + \alpha_3)q - \alpha_4t], \end{aligned}$$

where $q := q_n, \hat{q} := q_{n+1}$. From the definitions (5.14,5.15) this result can be readily recast as (5.16). The second (5.17) follows from a computation for $f_n + f_{n-1}$ using the shift operator L_{14} . \square

Remark 5.1. The two systems of recurrences (5.5,5.6) and (5.16,5.17) are related by an element of the S_4 subgroup of the $W_a(F_4)$ transformations, namely the generator x^3 [43]. This has the action

$$(5.18) \quad x^3 : \alpha_0 \leftrightarrow \alpha_4, t \mapsto \frac{t}{t-1}, q \mapsto \frac{t-q}{t-1}, p \mapsto -(t-1)p,$$

and when applying these transformations to (5.3), (5.4), (5.5), (5.6) we recover (5.14), (5.15), (5.16), (5.17) respectively.

Proposition 5.2. *Let $\{g_N\}_{N=0,1,\dots}, \{f_N\}_{N=0,1,\dots}$ satisfy the discrete Painlevé coupled difference equations associated with the degeneration of the rational surface $D_4^{(1)} \rightarrow D_5^{(1)}$*

$$(5.19) \quad g_{N+1}g_N = t \frac{(f_N + N + 1)(f_N + N + \mu + \omega)}{f_N(f_N - \mu - \bar{\omega})}$$

$$(5.20) \quad f_N + f_{N-1} = \mu + \bar{\omega} + \frac{N + 2\mu}{g_N - 1} + \frac{(N + 2\omega_1)t}{g_N - t},$$

where $t = e^{i\phi}$ subject to the initial conditions

$$g_0 = \frac{q_0 - t}{q_0 - 1}, \quad f_0 = \frac{1}{1-t} \left[(\mu + \omega)(q_0 - 1) + (\mu + \bar{\omega})(q_0 - t) - 2\omega_1 \frac{(q_0 - t)(q_0 - 1)}{q_0} \right]$$

with

$$(5.21) \quad q_0 = \frac{\omega_1}{\mu} \frac{-i \frac{d}{d\phi} \log e^{i\mu\phi} T_1(e^{i\phi})}{\mu + \omega + i \frac{d}{d\phi} \log e^{i\mu\phi} T_1(e^{i\phi})}.$$

Define $\{q_N, p_N\}_{N=0,1,\dots}$ in terms of $\{f_N, g_N\}_{N=0,1,\dots}$ by

$$(5.22) \quad q_N = \frac{g_N - t}{g_N - 1},$$

$$(5.23) \quad p_N = \frac{g_N - 1}{(1-t)g_N} \left[(g_N - 1)f_N - (\mu + \bar{\omega})g_N + (N + 2\omega_1) \frac{(1-t)g_N}{g_N - t} - N - \mu - \omega \right].$$

Then with $T_0(e^{i\phi}) = 1$ and $T_1(e^{i\phi}) = w_0(e^{i\phi})$ as given by (4.2,4.3), $\{T_N\}_{N=2,3,\dots}$ is specified by the recurrence

$$(5.24) \quad - (N + \mu + \omega)(N + \mu + \bar{\omega}) \frac{T_{N+1}T_{N-1}}{T_N^2} \\ = q_N(q_N - 1)^2 p_N^2 + [(2\mu - N)q_N + N + 2\omega_1](q_N - 1)p_N - 2\mu N q_N - N(N + 2\omega_1).$$

Proof. Let $Y_n := L_{14}^{-1}K_n - K_n = K_{n+1} - K_n$ and from Table 1 of [22] we have

$$Y_n = -\frac{(t-1)q_n}{q_n-1} \left\{ (q_n-1)p_n + \alpha_0 + \alpha_2 - 1 + \frac{(1-\alpha_0-\alpha_2)(\alpha_1+\alpha_2+\alpha_3)}{q_n(q_n-1)p_n + (\alpha_1+\alpha_2)q_n + \alpha_4} \right\}.$$

Now consider

$$\begin{aligned} t(t-1) \frac{d}{dt} \log \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} &= K_{n+1} + K_{n-1} - 2K_n \\ &= Y_n - L_{14}Y_n, \end{aligned}$$

and this latter difference, upon again consulting Table 1 of [22], turns out to be

$$\begin{aligned} Y_n - L_{14}Y_n &= \\ t(t-1) \frac{d}{dt} \log [q_n(q_n-1)^2 p_n^2 + [(\alpha_1+2\alpha_2)q_n + \alpha_4](q_n-1)p_n + \alpha_2[(\alpha_1+\alpha_2)q_n + \alpha_4]] &. \end{aligned}$$

After integrating both expressions and introducing an integration constant Eq. (5.24) follows. \square

We now seek to relate the results of the τ -function approach to the theory developed for the orthogonal polynomials on the unit circle with semi-classical weights as given in the previous section. However we will only discuss the scheme given in Proposition 5.1 as this is the simplest.

Proposition 5.3. *The transformations linking the Hamiltonian variables q_N, p_N in Proposition 5.1 to the reflection coefficients r_N, \bar{r}_N for the system of orthogonal polynomials with the weight (4.1) are given implicitly by*

$$\begin{aligned} (5.25) \quad q_N p_N + \mu + \bar{\omega} &= \frac{(N + \mu + \bar{\omega})r_N \bar{r}_N}{(N + \mu + \bar{\omega})r_N \bar{r}_N - \mu + \omega} \frac{1}{q_N - 1} \\ &\times \left[(N + 2\omega_1)(q_N - 1) - t \frac{l_N}{\kappa_N} + Nt + (N + 1 + \mu + \bar{\omega})(1 - r_N \bar{r}_N) t \frac{r_{N+1}}{r_N} \right] \\ &= (N + \mu + \bar{\omega})[(N + \mu + \omega)r_N \bar{r}_N - \mu + \bar{\omega}] \end{aligned}$$

$$\begin{aligned} (5.26) \quad &\times \frac{q_N}{(N + 2\omega_1)q_N + t \frac{l_N}{\kappa_N} - Nt - (N - 1 + \mu + \bar{\omega})(1 - r_N \bar{r}_N) t \frac{\bar{r}_{N-1}}{\bar{r}_N}}, \\ (q_N - 1)p_N + \mu + \omega &= (N + \mu + \omega)[(N + \mu + \bar{\omega})r_N \bar{r}_N - \mu + \omega] \end{aligned}$$

$$\begin{aligned} (5.27) \quad &\times \frac{q_N - 1}{(N + 2\omega_1)(q_N - 1) - t \frac{l_N}{\kappa_N} + Nt + (N + 1 + \mu + \bar{\omega})(1 - r_N \bar{r}_N) t \frac{r_{N+1}}{r_N}} \\ &= \frac{(N + \mu + \omega)r_N \bar{r}_N}{(N + \mu + \omega)r_N \bar{r}_N - \mu + \bar{\omega}} \frac{1}{q_N} \end{aligned}$$

$$(5.28) \quad \times \left[(N + 2\omega_1)q_N + t \frac{l_N}{\kappa_N} - Nt - (N - 1 + \mu + \bar{\omega})(1 - r_N \bar{r}_N) \frac{\bar{r}_{N-1}}{\bar{r}_N} \right].$$

Proof. We require in addition to the primary shift operator L_{01}^{-1} generating the $N \mapsto N+1$ sequence another operator which has the action $i\omega_2 \mapsto i\omega_2 - 1$. This is the secondary shift operator $T_{34}^{-1} = r_1 s_4 s_2 s_0 s_1 s_2 s_4$ and has the action $T_{34}^{-1} : \alpha_3 \rightarrow \alpha_3 + 1, \alpha_4 \rightarrow \alpha_4 - 1$. From

Table 1 of [22] we compute the actions of T_{34}^{-1}, T_{34} on the Hamiltonian to be

$$\begin{aligned} T_{34}^{-1} \cdot K_n - K_n &= -q_n(q_n - 1)p_n \\ &\quad + (\alpha_0 + \alpha_4 - 1)(q_n - 1) - (\alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3) \frac{q_n - 1}{(q_n - 1)p_n - \alpha_3} \\ T_{34} \cdot K_n - K_n &= -q_n(q_n - 1)p_n \\ &\quad + (\alpha_0 + \alpha_3 - 1)q_n - (\alpha_2 + \alpha_4)(\alpha_1 + \alpha_2 + \alpha_4) \frac{q_n}{q_n p_n - \alpha_4} \end{aligned}$$

However

$$T_{34}^{-1} \cdot K_n - K_n = t(t-1) \frac{d}{dt} \log \frac{I_n^1}{I_n^0} = t(t-1) \frac{d}{dt} \log r_n$$

and we employ the results of Corollary 3.2 and the evaluation of the coefficient functions in (4.8,4.9) to arrive at

$$\begin{aligned} (t-1) \frac{\dot{r}_N}{r_N} &= \frac{l_N}{\kappa_N} - N - (N+1+\mu+\bar{\omega})(1-r_N \bar{r}_N) \frac{r_{N+1}}{r_N}, \\ (t-1) \frac{\dot{\bar{r}}_N}{\bar{r}_N} &= -\frac{l_N}{\kappa_N} + N + (N-1+\mu+\bar{\omega})(1-r_N \bar{r}_N) \frac{\bar{r}_{N-1}}{\bar{r}_N}. \end{aligned}$$

In addition we note that after recalling (2.9), (5.12) factorises into

$$(N+\mu+\omega)(N+\mu+\bar{\omega})r_N \bar{r}_N = [q_N p_N + \mu + \bar{\omega}][(q_N - 1)p_N + \mu + \omega].$$

The stated results, (5.25-5.28), then follow. \square

6. APPLICATIONS TO PHYSICAL MODELS

6.1. Random Matrix Averages. A specialisation of the above results with great interest in the application of random matrices [32] is the quantity

$$(6.1) \quad F_N^{\text{CUE}}(u; \mu) := \left\langle \prod_{l=1}^N |u + z_l|^{2\mu} \right\rangle_{\text{CUE}_N}.$$

This has the interpretation as the average of the 2μ -th power of the absolute value of the characteristic polynomial for the CUE. In the case $|u| = 1$ (6.1) is independent of u and has the well-known (see e.g. [8]) Gamma function evaluation

$$(6.2) \quad \left\langle \prod_{l=1}^N |u + z_l|^{2\mu} \right\rangle_{\text{CUE}_N} \Big|_{u=e^{i\phi}} = \left\langle \prod_{l=1}^N |1 + z_l|^{2\mu} \right\rangle_{\text{CUE}_N} = \prod_{j=0}^{N-1} \frac{j! \Gamma(j+1+2\mu)}{\Gamma^2(j+1+\mu)},$$

when $\Re(\mu) > -1/2$. For $|u| < 1$ we see by an appropriate change of variables that

$$(6.3) \quad \left\langle \prod_{l=1}^N |u + z_l|^{2\mu} \right\rangle_{\text{CUE}_N} = \left\langle \prod_{l=1}^N (1 + |u|^2 z_l)^\mu (1 + 1/z_l)^\mu \right\rangle_{\text{CUE}_N}$$

$$(6.4) \quad = {}_2F_1^{(1)}(-\mu, -\mu; N; t_1, \dots, t_N) \Big|_{t_1=\dots=t_N=|u|^2},$$

where the second equality follows from (4.49). For $|u| > 1$ we can use the simple functional equation

$$(6.5) \quad \left\langle \prod_{l=1}^N |u + z_l|^{2\mu} \right\rangle_{\text{CUE}_N} = |u|^{2\mu N} \left\langle \prod_{l=1}^N \left| \frac{1}{u} + z_l \right|^{2\mu} \right\rangle_{\text{CUE}_N}$$

to relate this case back to the case $|u| < 1$.

The weight in the first equality of (6.4) is a special case of (4.1). In terms of the parameters of the form (4.1) we observe that $\xi = 0$, $2\mu \mapsto \mu$, $\omega = \bar{\omega} = \mu/2$, i.e. $\omega_2 = 0$ and $t = |u|^2$. The trigonometric moments are

$$(6.6) \quad w_{-n} = \frac{\Gamma(\mu+1)}{n!\Gamma(\mu+1-n)} {}_2F_1(-\mu, -\mu+n; n+1; |u|^2) \quad n \in \mathbb{Z}_{\geq 0}$$

$$(6.7) \quad w_n = |u|^{2n} w_{-n} \quad n \in \mathbb{Z}_{\geq 0}.$$

The results of Section 4 then allow (6.1) to be computed by a recurrence involving the corresponding reflection coefficients.

Corollary 6.1. *The general moments of the characteristic polynomial $|\det(u+U)|$ for arbitrary exponent 2μ with respect to the finite CUE ensemble $U \in U(N)$ of rank N is given by the system of recurrences*

$$(6.8) \quad \frac{F_{N+1}^{\text{CUE}} F_{N-1}^{\text{CUE}}}{(F_N^{\text{CUE}})^2} = 1 - |u|^{2N} r_N^2,$$

with initial values

$$(6.9) \quad F_0^{\text{CUE}} = 1, \quad F_1^{\text{CUE}} = {}_2F_1(-\mu, -\mu; 1; |u|^2),$$

and the recurrence relation for the reflection coefficient r_N

$$(6.10) \quad \begin{aligned} 2|u|^{2N} r_N r_{N-1} - |u|^2 - 1 &= \frac{1 - |u|^{2N} r_N^2}{r_N} [(N+1+\mu)|u|^2 r_{N+1} + (N-1+\mu)r_{N-1}] \\ &\quad - \frac{1 - |u|^{2(N-1)} r_{N-1}^2}{r_{N-1}} [(N+\mu)|u|^2 r_N + (N-2+\mu)r_{N-2}], \end{aligned}$$

subject to the initial values

$$(6.11) \quad r_0 = 1, \quad r_1 = -\mu \frac{{}_2F_1(-\mu, -\mu+1; 2; |u|^2)}{{}_2F_1(-\mu, -\mu; 1; |u|^2)}.$$

Proof. From either (4.10), (4.11) or (4.35) and the fact that $\bar{r}_1 = |u|^2 r_1$ we can repeat the arguments of Corollary 4.4 to deduce that $\bar{r}_N = |u|^{2N} r_N$ for $N \geq 0$. The recurrence relation follows simply from the specialisation of (4.20) and the initial conditions from the $N = 1$ case. \square

Another spectral statistic of fundamental importance in random matrix theory is the gap probability for the circular unitary ensembles, and this is the specialisation whereby $\mu = \omega = \bar{\omega} = 0$, $|t| = 1$ so the angle $\phi \in [0, 2\pi)$, whilst $\xi \in \mathbb{C}$ is general (one is mainly interested in an open neighbourhood of $\xi = 1$). The generating function for the probability of finding exactly k eigenvalues $z = e^{i\theta}$ within the sector of the unit circle $\theta \in (\pi - \phi, \pi]$ is denoted by $E_N^{\text{CUE}}((0, \phi); \xi)$ and has the definition

$$(6.12) \quad E_N^{\text{CUE}}((0, \phi); \xi) := \frac{1}{C_N} \left(\int_{-\pi}^{\pi} -\xi \int_{\pi-\phi}^{\pi} \right) d\theta_1 \dots \left(\int_{-\pi}^{\pi} -\xi \int_{\pi-\phi}^{\pi} \right) d\theta_N \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2,$$

where the normalisation $C_N = (2\pi)^N N!$. It is well known that Toeplitz elements with such a symbol have the form

$$(6.13) \quad w_n = \delta_{n,0} + \frac{\xi}{2\pi i} (-1)^{n+1} \frac{t^n - 1}{n},$$

which is easily recovered from the general expression (4.2). A recurrence scheme for the generating function (6.12) involving particular examples of the coupled discrete Painlevé equations (1.8), (1.9) has been presented in [20]. Here we use recurrences found herein for r_N , \bar{r}_N , together with the fact that for (6.12) one has $r_N = t^{-N}\bar{r}_N$, to replace the role of the coupled recurrences from [20] by a single recurrence.

Corollary 6.2. *The generating function for the probability of finding exactly k eigenvalues $z = e^{i\theta}$ from the ensemble of random $N \times N$ unitary matrices within the sector of the unit circle $\theta \in (\pi - \phi, \pi]$ is given by the following system of recurrences in the rank of the ensemble N ,*

$$(6.14) \quad \frac{E_{N+1}^{\text{CUE}} E_{N-1}^{\text{CUE}}}{(E_N^{\text{CUE}})^2} = 1 - x_N^2,$$

where the initial values are

$$(6.15) \quad E_0^{\text{CUE}} = 1, \quad E_1^{\text{CUE}} = 1 - \frac{\xi}{2\pi}\phi,$$

and the auxiliary variables x_N are determined by the quasi-linear third order recurrence relation

$$(6.16) \quad 2x_N x_{N-1} - 2 \cos \frac{\phi}{2} = \frac{1 - x_N^2}{x_N} [(N+1)x_{N+1} + (N-1)x_{N-1}] \\ - \frac{1 - x_{N-1}^2}{x_{N-1}} [Nx_N + (N-2)x_{N-2}],$$

or the quadratic second order recurrence relation

$$(6.17) \quad (1 - x_N^2)^2 [(N+1)^2 x_{N+1}^2 + (N-1)^2 x_{N-1}^2] + 2(N^2 - 1)(1 - x_N^4)x_{N+1}x_{N-1} \\ + 4N \cos \frac{\phi}{2} x_N (1 - x_N^2) [(N+1)x_{N+1} + (N-1)x_{N-1}] + 4N^2 x_N^2 \left[\cos^2 \frac{\phi}{2} - x_N^2 \right] = 0,$$

along with the initial values

$$(6.18) \quad x_{-1} = 0, \quad x_0 = 1, \quad x_1 = -\frac{\xi}{\pi} \frac{\sin \frac{\phi}{2}}{1 - \frac{\xi}{2\pi}\phi}.$$

Proof. The first recurrence relation follows directly from the general recurrence (4.20) and Corollary 4.4 whilst the second follows from (4.29). \square

6.2. 2-D Ising Model. It has been known for some time that the diagonal spin correlations in the square lattice Ising model could be evaluated in terms of the Painlevé sixth transcendent [29] and in this work a coupled system of difference equations involving eleven variables were given. Here we give what we consider to be the simplest set of recurrence relations for these correlations as a special case of the general theory above. The diagonal correlation functions are given by [42]

$$(6.19) \quad \langle \sigma_{0,0} \sigma_{N,N} \rangle = \begin{cases} \det(a_{i-j}(k))_{1 \leq i,j \leq N} & \text{if } k > 1 \text{ or } T < T_c, \\ \det(\tilde{a}_{i-j}(k))_{1 \leq i,j \leq N} & \text{if } k < 1 \text{ or } T > T_c, \end{cases}$$

where

$$(6.20) \quad a_n(k) := \frac{1}{2\pi i} \int_{\mathbb{T}} d\zeta \zeta^n \sqrt{\frac{k\zeta^{-1} - 1}{k\zeta - 1}}, \quad \tilde{a}_n(k) := \frac{1}{2\pi i} \int_{\mathbb{T}} d\zeta \zeta^{n-1} \sqrt{\frac{1 - k\zeta}{1 - k\zeta^{-1}}},$$

and the argument is defined $k = \sinh^2(2J/k_B T)$, J being the coupling strength and T the temperature. The weight appearing in the Toeplitz determinant form of the low temperature correlation is

$$(6.21) \quad w(z) = C z^{1/4} |1+z|^{-1/2} (1+k^{-2}z)^{1/2} = C z^{1/2} (1+z)^{-1/2} (1+k^{-2}z)^{1/2},$$

which is a special case of (4.1) with $\xi = 0$, $\mu = 1/4$, $\omega_1 = -1/4$, $\omega_2 = i/2$ and $t = 1/k^2$. Here $a_n = (-1)^n k^{-n} w_{-n}(1/k^2)$. In the high temperature regime the exponents μ, ω_1 are sign reversed and $t = k^2$, so $\tilde{a}_n = (-1)^n k^n w_{-n}(k^2)$. So these cases form an interesting example where $\omega_2 \neq 0$, so both r_N, \bar{r}_N are distinct and independent in contrast to the random matrix and quantum many-body cases. As is also well known the Toeplitz matrix elements in the low temperature regime are given by

$$(6.22) \quad w_{-n} = \frac{(-1)^n}{\pi} \frac{\Gamma(n+1/2)\Gamma(1/2)}{\Gamma(n+1)} {}_2F_1(-1/2, n+1/2; n+1; k^{-2}), \quad n \geq 0,$$

$$(6.23) \quad w_n = \frac{(-1)^{n+1} k^{-2n}}{\pi} \frac{\Gamma(n-1/2)\Gamma(3/2)}{\Gamma(n+1)} {}_2F_1(1/2, n-1/2; n+1; k^{-2}), \quad n > 0,$$

whilst those in the high temperature regime are

$$(6.24) \quad w_{-n} = \frac{(-1)^n k^{2n+1}}{\pi} \frac{\Gamma(n+1/2)\Gamma(3/2)}{\Gamma(n+2)} {}_2F_1(1/2, n+1/2; n+2; k^2), \quad n \geq 0,$$

$$(6.25) \quad w_n = \frac{(-1)^{n-1}}{\pi k} \frac{\Gamma(n-1/2)\Gamma(1/2)}{\Gamma(n)} {}_2F_1(-1/2, n-1/2; n; k^2), \quad n > 0.$$

Corollary 6.3. *The diagonal correlation function for the Ising model valid in both the low and high temperature phases (with $k \mapsto 1/k$ in the latter case) is determined by*

$$(6.26) \quad \frac{\langle \sigma_{0,0} \sigma_{N+1,N+1} \rangle \langle \sigma_{0,0} \sigma_{N-1,N-1} \rangle}{\langle \sigma_{0,0} \sigma_{N,N} \rangle^2} = 1 - r_N \bar{r}_N,$$

along with the quasi-linear 2/1

$$(6.27) \quad (2N+3)k^{-2}(1-r_N \bar{r}_N)r_{N+1} + 2N[k^{-2} + 1 - (2N-1)k^{-2}r_N \bar{r}_{N-1}]r_N \\ + (2N-3)[(2N-1)r_N \bar{r}_N + 1]r_{N-1} = 0,$$

and 1/2 recurrence relations

$$(6.28) \quad (2N+1)(1-r_N \bar{r}_N)\bar{r}_{N+1} + 2N[(2N-3)\bar{r}_N r_{N-1} + k^{-2} + 1]\bar{r}_N \\ + (2N-1)k^{-2}[-(2N+1)r_N \bar{r}_N + 1]\bar{r}_{N-1} = 0,$$

subject to initial conditions for the low temperature regime

$$(6.29) \quad r_0 = 1, \quad \bar{r}_0 = 1, \quad r_1 = \frac{2-k^2}{3} + \frac{k^2-1}{3} \frac{K(k^{-1})}{E(k^{-1})}, \quad \bar{r}_1 = -1 + \frac{k^2-1}{k^2} \frac{K(k^{-1})}{E(k^{-1})},$$

or to the initial conditions for the high temperature regime given by

$$(6.30) \quad r_0 = 1, \quad \bar{r}_0 = 1, \quad r_1 = \frac{1}{3} \left\{ \frac{2}{k^2} - \frac{E(k)}{(k^2-1)K(k) + E(k)} \right\}, \quad \bar{r}_1 = -\frac{k^2 E(k)}{(k^2-1)K(k) + E(k)},$$

where $K(k), E(k)$ are the complete elliptic integrals of the first and second kind respectively.

Proof. (6.27, 6.28) follow from (4.35) and its "conjugate" upon the specialisation to the parameters above. The initial conditions follow from explicit evaluation of the Toeplitz determinants. \square

The correlation function and reflection coefficients have particularly simple, yet general forms when expressed in terms of generalised hypergeometric functions.

Corollary 6.4. *In the low temperature phase the diagonal correlation function is given by*

$$(6.31) \quad \langle \sigma_{0,0} \sigma_{N,N} \rangle = {}_2F_1^{(1)}(-1/2, 1/2; N; t_1, \dots, t_N) \Big|_{t_1=\dots=t_N=1/k^2},$$

whilst the reflection coefficients are given by

$$(6.32) \quad r_N = (-1)^N \frac{(-1/2)_N}{N!} \frac{{}_2F_1^{(1)}(-1/2, 3/2; N+1; t_1, \dots, t_N)}{{}_2F_1^{(1)}(-1/2, 1/2; N; t_1, \dots, t_N)} \Big|_{t_1=\dots=t_N=1/k^2},$$

$$(6.33) \quad \bar{r}_N = (-1)^N \frac{(N-1)!}{(1/2)_N} \frac{\lim_{\epsilon \rightarrow 0} \epsilon {}_2F_1^{(1)}(-1/2, -1/2; N-1+\epsilon; t_1, \dots, t_N)}{{}_2F_1^{(1)}(-1/2, 1/2; N; t_1, \dots, t_N)} \Big|_{t_1=\dots=t_N=1/k^2}.$$

In the high temperature phase the diagonal correlation function is

$$(6.34) \quad \langle \sigma_{0,0} \sigma_{N,N} \rangle = \frac{(2N-1)!!}{2^N N!} k^N {}_2F_1^{(1)}(1/2, 1/2; N+1; t_1, \dots, t_N) \Big|_{t_1=\dots=t_N=k^2},$$

and the reflection coefficients are given by

$$(6.35) \quad r_N = (-1)^N \frac{(-1/2)_N}{(N+1)!} \frac{{}_2F_1^{(1)}(1/2, 3/2; N+2; t_1, \dots, t_N)}{{}_2F_1^{(1)}(1/2, 1/2; N+1; t_1, \dots, t_N)} \Big|_{t_1=\dots=t_N=k^2},$$

$$(6.36) \quad \bar{r}_N = (-1)^N \frac{N!}{(1/2)_N} \frac{{}_2F_1^{(1)}(1/2, -1/2; N; t_1, \dots, t_N)}{{}_2F_1^{(1)}(1/2, 1/2; N+1; t_1, \dots, t_N)} \Big|_{t_1=\dots=t_N=k^2}.$$

Proof. The evaluations in the low temperature phase follow from (4.50,4.51), although some care needs to be taken with \bar{r}_N because $-\mu + \bar{\omega} = 0$. The limit that arises has a series development

$$(6.37) \quad \lim_{\epsilon \rightarrow 0} \epsilon {}_2F_1^{(1)}(-1/2, -1/2; N-1+\epsilon; t_1, \dots, t_N) = \sum_{\kappa: l(\kappa)=N}^{\infty} \frac{([-1/2]_{\kappa}^{(1)})^2}{[N]_{\kappa}^{(1)}} \frac{\prod_{j=1}^N (N-j+\kappa_j)}{(N-1)!} \frac{s_{\kappa}(t_1, \dots, t_N)}{h_{\kappa}}$$

so that only those terms with lengths $l(\kappa) = N$ contribute to the sum. The high temperature expressions follow from the low temperature ones through the transformation $\mu \leftrightarrow \omega_1$. \square

It is of interest to note that as N grows more of the leading order terms in the expansion of (6.31) become independent of N , and the following limit becomes explicit

$$(6.38) \quad \lim_{N \rightarrow \infty} \langle \sigma_{0,0} \sigma_{N,N} \rangle = (1 - k^{-2})^{1/4}.$$

At zero temperature, $k = \infty$, the solutions simplify to

$$(6.39) \quad r_N = (-1)^N \frac{(-1/2)_N}{N!}, \quad \bar{r}_N = 0 \quad (N \geq 1), \quad \langle \sigma_{0,0} \sigma_{N,N} \rangle = 1,$$

whilst at the critical point, $k = 1$, we have the simple solutions

$$(6.40) \quad r_N = \frac{(-1)^{N-1}}{(2N+1)(2N-1)}, \quad \bar{r}_N = (-1)^N, \quad l_N = \frac{N}{2N+1},$$

$$(6.41) \quad \langle \sigma_{0,0} \sigma_{N,N} \rangle = \prod_{j=1}^N \frac{\Gamma^2(j)}{\Gamma(j+1/2)\Gamma(j-1/2)},$$

and at infinite temperature they become

$$(6.42) \quad r_N = (-1)^N \frac{(-1/2)_N}{(N+1)!}, \quad \bar{r}_N = (-1)^N \frac{N!}{(1/2)_N}, \quad \langle \sigma_{0,0} \sigma_{N,N} \rangle = 0 \quad (N \geq 1),$$

in agreement with the known results [42].

Acknowledgments. This research has been supported by the Australian Research Council. NSW appreciates the generosity of Will Orrick in supplying expansions of the Toeplitz determinants for the diagonal correlations of the Ising model and the assistance of Paul Leopardi in calculating gap probabilities for the CUE. Our manuscript has benefited from the critical reading by Alphonse Magnus and we thank him, Mourad Ismail and Percy Deift for their advice and suggestions.

REFERENCES

1. *Oeuvres de Laguerre. Tome I*, Chelsea Publishing Co., Bronx, N.Y., 1972, Algèbre. Calcul intégral, Rédigées par Ch. Hermite, H. Poincaré et E. Rouché, Réimpression de l'édition de 1898. MR 52 #13292
2. M. Adler and P. van Moerbeke, *Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice*, **math-ph/0201063**, 2002.
3. J. Baik, *Riemann-Hilbert problems for last passage percolation*, **PR/0107079**, 2001.
4. J. Baik, P. Deift, and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), no. 4, 1119–1178. MR **2000e:05006**
5. ———, *On the distribution of the length of the second row of a Young diagram under Plancherel measure*, Geom. Funct. Anal. **10** (2000), no. 4, 702–731. MR **2001m:05258a**
6. J. Baik, P. Deift, K.T.-R. McLaughlin, P. Miller, and X. Zhou, *Optimal tail estimates for directed last passage site percolation with geometric random variables*, Adv. Theor. Math. Phys. **5** (2001), no. 6, 1207–1250. MR **1 926 668**
7. J. Baik and E. M. Rains, *Algebraic aspects of increasing subsequences*, Duke Math. J. **109** (2001), no. 1, 1–65. MR **1 844 203**
8. T. H. Baker and P. J. Forrester, *The Calogero-Sutherland model and generalized classical polynomials*, Comm. Math. Phys. **188** (1997), no. 1, 175–216. MR **99c:33012**
9. W. C. Bauldry, *Estimates of asymmetric Freud polynomials on the real line*, J. Approx. Theory **63** (1990), no. 2, 225–237. MR **92c:33008**
10. S. Belmehdi and A. Ronveaux, *Laguerre-Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Approx. Theory **76** (1994), no. 3, 351–368. MR **95f:42038**
11. S. S. Bonan and D. S. Clark, *Estimates of the Hermite and the Freud polynomials*, J. Approx. Theory **63** (1990), no. 2, 210–224. MR **92c:33007**
12. A. Borodin, *Discrete gap probabilities and discrete Painlevé equations*, **math-ph/0111008**, 2001.

13. A. Borodin and D. Boyarchenko, *Distribution of the first particle in discrete orthogonal polynomial ensembles*, **math-ph/0204001**, 2002.
14. A. Borodin and G. Olshanski, *Distributions on partitions, point processes, and the hypergeometric kernel*, *Comm. Math. Phys.* **211** (2000), no. 2, 335–358. MR **2001k**:33031
15. P. A. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, New York University Courant Institute of Mathematical Sciences, New York, 1999. MR **2000g**:47048
16. A. S. Fokas, A. R. It's, and A. V. Kitaev, *Discrete Painlevé equations and their appearance in quantum gravity*, *Comm. Math. Phys.* **142** (1991), no. 2, 313–344. MR **93a**:58080
17. ———, *The isomonodromy approach to matrix models in 2D quantum gravity*, *Comm. Math. Phys.* **147** (1992), no. 2, 395–430. MR **93h**:81115
18. P. J. Forrester, *Log Gases and Random Matrices*,
<http://www.ms.unimelb.edu.au/matpjf/matpjf.html>.
19. P. J. Forrester, N. E. Frankel, T. Garoni, and N. S. Witte, *Painlevé transcendent evaluations of finite system density matrices for 1d impenetrable bosons*, **math-ph/0207005** to appear *Commun. Math. Phys.*, 2003.
20. P. J. Forrester and N. S. Witte, *Discrete Painlevé equations and random matrix averages*, **math-ph/0304020**.
21. ———, *Discrete Painlevé Equations, Orthogonal Polynomials on the Unit Circle and n -recurrences for averages over $U(N)$ – P_{III} and P_V τ -functions*, **math-ph/0305029**.
22. ———, *Application of the τ -function theory of Painlevé equations to random matrices: PVI, the JUE, CyUE, cJUE and scaled limits*, **math-ph/0204008** to appear *Nagoya Math. J.*, 2003.
23. Géza Freud, *On the coefficients in the recursion formulae of orthogonal polynomials*, *Proc. Roy. Irish Acad. Sect. A* **76** (1976), no. 1, 1–6. MR **54** #7913
24. Ya. L. Geronimus, *Orthogonal polynomials: Estimates, asymptotic formulas, and series of polynomials orthogonal on the unit circle and on an interval*, Authorized translation from the Russian, Consultants Bureau, New York, 1961. MR **24** #A3469
25. ———, *Polynomials orthogonal on a circle and their applications*, *Translations, Ser. 1, Vol. 3: Series and approximation*, vol. 3, American Mathematical Society, Providence, R.I., 1962, p. 78.
26. ———, *Orthogonal polynomials*, *Amer. Math. Soc. Transl. Series 2*, vol. 108, American Mathematical Society, Providence, Rhode Island, 1977.
27. M. E. H. Ismail and N. S. Witte, *Discriminants and functional equations for polynomials orthogonal on the unit circle*, *J. Approx. Theory* **110** (2001), no. 2, 200–228. MR **2002e**:33011
28. A. R. It's, A. V. Kitaev, and A. S. Fokas, *Matrix models of two-dimensional quantum gravity, and isomonodromic solutions of Painlevé “discrete equations”*, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **187** (1991), no. *Differentsialnaya Geom. Gruppy Li i Mekh.* **12**, 3–30, 171, 174. MR **93e**:81114
29. M. Jimbo and T. Miwa, *Studies on holonomic quantum fields. XVII*, *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), no. 9, 405–410. MR **85h**:82016
30. W. B. Jones, O. Njåstad, and W. J. Thron, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, *Bull. London Math. Soc.* **21** (1989), no. 2, 113–152. MR **90e**:42027
31. J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, *SIAM J. Math. Anal.* **24** (1993), no. 4, 1086–1110. MR **94h**:33010
32. J. P. Keating and N. C. Snaith, *Random matrix theory and $\zeta(1/2 + it)$* , *Comm. Math. Phys.* **214** (2000), no. 1, 57–89. MR **2002c**:11107

33. F. Klein, *Vorlesungen über die Hypergeometrische Function*, Verlag von Julius Springer, Inc., Berlin, 1933, Gehalten an der Universität Göttingen in Wintersemester 1893/94.
34. A. Lenard, *Momentum distribution in the ground state of the one-dimensional system of impenetrable bosons*, J. Math. Phys. **5** (1964), no. 7, 930–943.
35. A. P. Magnus, *MAPA3072A Special topics in approximation theory 1999-2000: Semi-classical orthogonal polynomials on the unit circle*, <http://www.math.ucl.ac.be/~magnus/>.
36. ———, *Painlevé equations for semi-classical recurrence coefficients*, CA/9409228, 1994.
37. ———, *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications (Evian-Les-Bains, 1992), vol. 57, 1995, pp. 215–237. MR **96f**:42027
38. ———, *Freud's equations for orthogonal polynomials as discrete Painlevé equations*, Symmetries and integrability of difference equations (Canterbury, 1996), London Math. Soc. Lecture Note Ser., vol. 255, Cambridge Univ. Press, Cambridge, 1999, pp. 228–243. MR **2000k**:42036
39. F. Marcellán and I. A. Rocha, *On semiclassical linear functionals: integral representations*, Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications (Evian-Les-Bains, 1992), vol. 57, 1995, pp. 239–249. MR **96f**:42028
40. ———, *Complex path integral representation for semiclassical linear functionals*, J. Approx. Theory **94** (1998), no. 1, 107–127. MR **99h**:42047
41. P. Maroni, *Prolégomènes à l'étude des polynômes orthogonaux semi-classiques*, Ann. Mat. Pura Appl. (4) **149** (1987), 165–184. MR **89c**:33016
42. B. McCoy and T. T. Wu, *The Two-Dimensional Ising Model*, Harvard University Press, Harvard, 1973.
43. K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}* , Ann. Mat. Pura Appl. (4) **146** (1987), 337–381. MR **88m**:58062
44. H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229. MR **1** 882 403
45. G. Szegő, *Orthogonal polynomials*, third ed., Colloquium Publications **23**, American Mathematical Society, Providence, Rhode Island, 1967.
46. Z. M. Yan, *A class of generalized hypergeometric functions in several variables*, Canad. J. Math. **44** (1992), no. 6, 1317–1338. MR **94c**:33026